Millman's Formula in Data Fusion

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Abstract— **This paper deals with quantitative data fusion for linear stochastic dynamic systems. The stress is laid on the Millman's formula which is utilised for combining state estimates that are based on different data sets. Various filtering and smoothing problems are introduced and solved by using the same fusion principle. The optimal centralised multisensor filtering is compared with other filtering architectures by means of a numerical example.**

I. INTRODUCTION

The data fusion problem arises when there are multiple sources of data. This problem can be viewed quantitatively or qualitatively. The main tools of quantitative data fusion are statistics and probability theory. The qualitative point of view uses higher levels of abstraction. While the former approach deals with the object state represented by a vector of real numbers, which meaning can be e.g. object position and velocity, the latter uses symbols. Object types, relations among objects or more abstract inferences are the goals of qualitative data fusion.

There are several models of the data fusion process [1]. The most widely used is the JDL Data Fusion Model. It is a functional model which distinguishes several levels in the fusion process. The level 1 (Object Assessment) deals with objects and object states, the qualitative fusion is frequently used there, levels 2 and 3 (Situation and Impact Assessment) deals with quantitative fusion mainly. The artificial intelligence is the tool used within these levels. Level 4 is a metaprocess that optimises overall fusion performance. The levels are not organised hierarchically but they are bus-connected. The JDL model is only functional, the data fusion algorithms can combine more levels simultaneously.

Many areas of research exist. A general problem is the representation of uncertainty and handling it. Different approaches are suitable for each fusion level. The probability theory with Bayesian rule, fuzzy set theory or Dempster-Shafer theory serve as examples. More particular problems are multi-sensor multi-target tracking, condition-time monitoring, image fusion and classification or expert system design. In [2] a good overview of general data fusion approaches and techniques is given. The target tracking, which is the main qualitative fusion problem, can be found in [3].

Though fusion problems exists in one-sensor system, the crucial problems relate to multisensor systems. The term one-sensor system is usually understood as a system with only one estimator, not as a system with only one scalar measurement. The multisensor problems are divided into measurement-to-measurement, measurement-to-track and track-to-track fusion problems where the notation "track" comes from target-tracking domain and is equal to the notation "state estimate". This partition is closely coupled with the fusion architecture which can be centralised, hierarchical, decentralised or mixed. The connections among local estimators, presence of a central estimator, communicated data types, i.e. measurements or estimates, communication rates and other properties define the fusion architecture.

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The centralised architecture consists of one central processor which processes all measurements, no estimates are communicated. It corresponds to the standard problem formulation. In the hierarchical architecture, upper level estimators fuse estimates coming from lower level estimators, measurements are communicated to the lowest level estimators only, the communication structure is a tree. Bar-Shalom formula [4] can be used to fuse the estimates. The decentralised architecture does not have any top level estimator and there are not structure demands. For tree connected estimators the channel filters [5] can be used. The channel filters keep all information that is communicated between two correspondent nodes to prevent double counting of information. Other possible solution to the decentralised fusion problem is to use some upper bounds when fusing two estimates. This can be done by using the Covariance Intersection algorithm [6].

This paper deals with the quantitative data fusion. Measurements are generated by linear stochastic systems and prior information concerning the state of the system is available. Single state estimates are obtained from the measurements sets and from the prior information. The goal of the data fusion is to find an overall estimate that combines the single estimates according to the fusion architecture. A common fusion technique can be found for various smoothing and filtering problems.

This paper is a simulation study that compares various applications of the Millman's formula (MF) in fusion and estimation problems. The stress is laid on an alternative view on current estimation techniques as well.

The paper is organised as follows. The generalised Millman's formula is described in Section 2, a multisensor system is defined in Section 3. In Section 4, the use of the basic MF in the filtering and smoothing problems is shown. Multisensor fusion and hierarchical filtering which use the generalised Millman's formula are described in Section 5. A comparative example is performed in Section 6. Section 7 contains concluding remarks.

II. Generalised Millman's Formula

The Generalised Millman's Formula (GMF) is a tool for combination of two or more correlated and uncorrelated local estimates [7].

Let N local estimates of the vector $\mathbf{x} \in \mathbb{R}^{n_x}$ be supposed, i.e. $\hat{\mathbf{x}}_i$, $i = 1, 2 \ldots, N$, where $\mathbb{R}^{n_x}$ is n_x-dimensional Euclidean space. With the estimates, the covariance matrices of estimation error

$$\mathbf{P}_{ij} = \mathsf{cov}(\tilde{\mathbf{x}}_i, \tilde{\mathbf{x}}_j) \tag{1}$$

are associated, where $\tilde{\mathbf{x}}_i = \mathbf{x} - \hat{\mathbf{x}}_i$, and $i, j = 1, \ldots, N$.

The aim is to find the overall optimal linear estimate of $\mathbf{x}$ of the form

$$\hat{\mathbf{x}} = \sum_{i=1}^{N} \mathbf{c}_i \hat{\mathbf{x}}_i, \tag{2}$$

$$\sum_{i=1}^{N} \mathbf{c}_i = \mathbf{I}_{n_x}, \tag{3}$$

where $\mathbf{c}_i$ is the i-th $n_x \times n_x$ constant weighting matrix and $\mathbf{I}_{n_x}$ is the identity matrix of size n_x. The overall error covariance matrix $\mathbf{P} = \mathsf{cov}(\mathbf{x} - \hat{\mathbf{x}})$ is given by

$$\mathbf{P} = \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{c}_i \mathbf{P}_{ij} \mathbf{c}_j^T. \tag{4}$$

The GMF allows to find the weighting matrices $\mathbf{c}_i$ by minimisation of the following mean square error criterion

$$J(\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_N) = E\left(\left\| \mathbf{x} - \sum_{i=1}^{N} \mathbf{c}_i \hat{\mathbf{x}}_i \right\|^2 \right), \tag{5}$$

which leads to the linear equations

$$\sum_{i=1}^{N-1} \mathbf{c}_i (\mathbf{P}_{ij} - \mathbf{P}_{iN}) + \mathbf{c}_N (\mathbf{P}_{Nj} - \mathbf{P}_{NN}) = \mathbf{0} \tag{6}$$

$$\sum_{i=1}^{N} \mathbf{c}_i = \mathbf{I}_{n_x}, \tag{7}$$

where $j = 1, 2, \ldots, N-1$ [7].

Relations (2), (4), (6), and (7) represent the generalised Millman's formula for $N > 2$ local estimates $\hat{\mathbf{x}}_i$. Note that further discussion concerning generalised Millman's formula can be found in [7].

In the particular case with $N = 2$ the generalised Millman's formula can be written in closed form as

- the *Bar-Shalom-Campo formula* for the optimal combination of two correlated estimates [8]

$$\begin{aligned}\hat{\mathbf{x}} &= \hat{\mathbf{x}}_1 + (\mathbf{P}_{11} - \mathbf{P}_{12})(\mathbf{P}_{11} + \mathbf{P}_{22} - \mathbf{P}_{12} - \mathbf{P}_{21})^{-1} \times \\ &\times (\hat{\mathbf{x}}_2 - \hat{\mathbf{x}}_1) = (\mathbf{P}_{22} - \mathbf{P}_{21})(\mathbf{P}_{11} + \mathbf{P}_{22} - \mathbf{P}_{12} - \mathbf{P}_{21})^{-1} \times \\ &\times \hat{\mathbf{x}}_1 + (\mathbf{P}_{11} - \mathbf{P}_{12})(\mathbf{P}_{11} + \mathbf{P}_{22} - \mathbf{P}_{12} - \mathbf{P}_{21})^{-1} \hat{\mathbf{x}}_2 = \\ &= \mathbf{c}_1 \hat{\mathbf{x}}_1 + \mathbf{c}_2 \hat{\mathbf{x}}_2, \end{aligned} \tag{8}$$

$$\begin{aligned}\mathbf{P} &= \mathbf{c}_1 \mathbf{P}_{11} \mathbf{c}_1^T + \mathbf{c}_1 \mathbf{P}_{12} \mathbf{c}_2^T + \mathbf{c}_2 \mathbf{P}_{21} \mathbf{c}_1^T + \mathbf{c}_2 \mathbf{P}_{22} \mathbf{c}_2^T = \\ &= \mathbf{P}_{11} - (\mathbf{P}_{11} - \mathbf{P}_{12})(\mathbf{P}_{11} + \mathbf{P}_{22} - \mathbf{P}_{12} - \mathbf{P}_{21})^{-1} \times \\ &\times (\mathbf{P}_{11} - \mathbf{P}_{21}), \end{aligned} \tag{9}$$

  where $\mathbf{c}_1 = (\mathbf{P}_{22} - \mathbf{P}_{21})(\mathbf{P}_{11} + \mathbf{P}_{22} - \mathbf{P}_{12} - \mathbf{P}_{21})^{-1}$ and $\mathbf{c}_2 = (\mathbf{P}_{11} - \mathbf{P}_{12})(\mathbf{P}_{11} + \mathbf{P}_{22} - \mathbf{P}_{12} - \mathbf{P}_{21})^{-1}$, and
- the *Millman's formula* for the optimal combination of two uncorrelated estimates, i.e. with $\mathbf{P}_{12} = \mathbf{P}_{21} = \mathbf{0}$, [9]

$$\begin{aligned}\hat{\mathbf{x}} &= \mathbf{P}_{22}(\mathbf{P}_{11} + \mathbf{P}_{22})^{-1} \hat{\mathbf{x}}_1 + \mathbf{P}_{11}(\mathbf{P}_{11} + \mathbf{P}_{22})^{-1} \hat{\mathbf{x}}_2 = \\ &= \mathbf{c}_1 \hat{\mathbf{x}}_1 + \mathbf{c}_2 \hat{\mathbf{x}}_2, \end{aligned} \tag{10}$$

$$\mathbf{P} = \mathbf{c}_1 \mathbf{P}_{11} \mathbf{c}_1^T + \mathbf{c}_2 \mathbf{P}_{22} \mathbf{c}_2^T = \mathbf{P}_{11}(\mathbf{P}_{11} + \mathbf{P}_{22})^{-1} \mathbf{P}_{22}, \tag{11}$$

  where $\mathbf{c}_1 = \mathbf{P}_{22}(\mathbf{P}_{11} + \mathbf{P}_{22})^{-1}$ and $\mathbf{c}_2 = \mathbf{P}_{11}(\mathbf{P}_{11} + \mathbf{P}_{22})^{-1}$.

Utilisation of the generalised Millman's formula and its particular cases in the state estimation will be discussed in the following sections.

III. System Definition and State Estimation

Let the linear discrete-time stochastic system be described by equations

$$\mathbf{x}_{k+1} = \mathbf{F}\mathbf{x}_k + \mathbf{G}\mathbf{w}_k, \tag{12}$$

$$\mathbf{z}_k^{(j)} = \mathbf{H}^{(j)} \mathbf{x}_k + \mathbf{v}_k^{(j)}, \tag{13}$$

where $\mathbf{F} \in \mathbb{R}^{n_x \times n_x}$, $\mathbf{H}^{(j)} \in \mathbb{R}^{n_z \times n_x}$, and $\mathbf{G} \in \mathbb{R}^{n_x \times n_w}$ are known matrices, $\mathbf{x}_k \in \mathbb{R}^{n_x}$ is the immeasurable system state, $\mathbf{z}_k^{(j)} \in \mathbb{R}^{n_z}$ is the measurement coming from j-th sensor, $k = 0, 1, \ldots$ is a time index, and $j = 1, \ldots, S$ is the sensor number. The variables $\mathbf{w}_k \in \mathbb{R}^{n_w}$ and $\mathbf{v}_k^{(j)} \in \mathbb{R}^{n_z}$ are the state and measurement Gaussian noises with zero mean and with known covariance matrices $\mathbf{Q}$, $\mathbf{R}^{(j)}$, respectively. Both noises are mutually independent and independent of the system initial state described by the Gaussian pdf $p(\mathbf{x}_0) = \mathcal{N}\{\mathbf{x}_0 : \bar{\mathbf{x}}_0, \mathbf{P}_0\}$. Note that the processes $\{\mathbf{v}_k^{(j)}\}$ need not to be mutually independent.

The aim of the state estimation is to find an estimate of the system state $\mathbf{x}_k$ providing that measurements ${}^l\mathbf{z}$ are given, ${}^l\mathbf{z} = [\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_l]$. The estimate is usually in the form of the conditional probability density function $p(\mathbf{x}_k | {}^l\mathbf{z})$ or at least in the form of two conditional moments, the mean $\hat{\mathbf{x}}_{k|l} = E(\mathbf{x}_k | {}^l\mathbf{z})$ and the covariance matrix $\mathbf{P}_{k|l} = \mathsf{cov}(\mathbf{x}_k | {}^l\mathbf{z})$. The general solution to the estimation problem is then given by the functional recursive relations known for the prediction ($k > l$), the filtering ($k = l$), and the smoothing ($k < l$) [9].

Note that the recursive relations can be exactly solved for the linear Gaussian system (12) and (13).

IV. Millman's Formula in Filtering and Smoothing

This section deals with the utilisation of the Millman's formula in filtering and smoothing. For the sake of simplicity, the system (12), (13) with one sensor is supposed, i.e. $S = 1$.

A. Millman's formula in filtering

The simplest use of the MF could be found by analysing the Kalman filter (KF) [9].

The measurement update (filtering) equation of the KF

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k (\mathbf{z}_k - \mathbf{H}\hat{\mathbf{x}}_{k|k-1}), \tag{14}$$

where $\mathbf{K}_k$ is the Kalman gain, can be interpreted as a fusion of two estimates, namely

- the prior information $\hat{\mathbf{x}}_{k|k-1}$, i.e. predictive estimate of the state $\mathbf{x}_k$ using measurements ${}^{k-1}\mathbf{z}$, with covariance matrix $\mathbf{P}_{k|k-1}$ and
- the maximum likelihood (ML) state estimate $\hat{\mathbf{x}}_k^{ML}$

$$\hat{\mathbf{x}}_k^{ML} = (\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{z}_k \tag{15}$$

with covariance matrix $\mathbf{P}_{ML} = (\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})^{-1}$, which is based on the measurement $\mathbf{z}_k$ only.

To show that, an alternative expression of update equation (14) will be used [9]

$$\hat{\mathbf{x}}_{k|k} = (\mathbf{I}_{n_x} - \mathbf{K}_k\mathbf{H})\hat{\mathbf{x}}_{k|k-1} + \mathbf{P}_{k|k}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{z}_k, \tag{16}$$

where the Kalman gain $\mathbf{K}_k$ and the posterior covariance matrix $\mathbf{P}_{k|k}$ are given by

$$\mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{H}^T(\mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^T + \mathbf{R})^{-1} = \mathbf{P}_{k|k}\mathbf{H}^T\mathbf{R}^{-1}, \tag{17}$$
$$\mathbf{P}_{k|k} = (\mathbf{P}_{k|k-1}^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})^{-1} = (\mathbf{I}_{n_x} - \mathbf{K}_k\mathbf{H})\mathbf{P}_{k|k-1}. \tag{18}$$

With respect to the relation $\mathbf{P}_{ML}^{-1}\hat{\mathbf{x}}_k^{ML} = \mathbf{H}^T\mathbf{R}^{-1}\mathbf{z}_k$ and (18), equation (16) can be written as

$$\hat{\mathbf{x}}_{k|k} = \mathbf{P}_{k|k}\mathbf{P}_{k|k-1}^{-1}\hat{\mathbf{x}}_{k|k-1} + \mathbf{P}_{k|k}\mathbf{P}_{ML}^{-1}\hat{\mathbf{x}}_k^{ML}. \tag{19}$$

As the particular state estimates $\hat{\mathbf{x}}_{k|k-1}$ and $\hat{\mathbf{x}}_k^{ML}$ are independent, the posterior state estimate $\hat{\mathbf{x}}_{k|k}$ (19) can be written in terms of Millman's formula (10), i.e. as

$$\hat{\mathbf{x}}_{k|k} = \mathbf{c}_1\hat{\mathbf{x}}_1 + \mathbf{c}_2\hat{\mathbf{x}}_2, \tag{20}$$

where $\hat{\mathbf{x}}_1 = \hat{\mathbf{x}}_{k|k-1}$, $\hat{\mathbf{x}}_2 = \hat{\mathbf{x}}_k^{ML}$,

$$\begin{aligned}\mathbf{c}_1 &= \mathbf{P}_{22}(\mathbf{P}_{11} + \mathbf{P}_{22})^{-1} = (\mathbf{P}_{11}^{-1} + \mathbf{P}_{22}^{-1})^{-1}\mathbf{P}_{11}^{-1} = \\ &= \left(\mathbf{P}_{k|k-1}^{-1} + (\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})\right)^{-1}\mathbf{P}_{k|k-1}^{-1} = \mathbf{P}_{k|k}\mathbf{P}_{k|k-1}^{-1}, \end{aligned} \tag{21}$$
$$\begin{aligned}\mathbf{c}_2 &= \mathbf{P}_{11}(\mathbf{P}_{11} + \mathbf{P}_{22})^{-1} = (\mathbf{P}_{11}^{-1} + \mathbf{P}_{22}^{-1})^{-1}\mathbf{P}_{22}^{-1} = \\ &= \left(\mathbf{P}_{k|k-1}^{-1} + (\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})\right)^{-1}\mathbf{P}_{ML}^{-1} = \mathbf{P}_{k|k}\mathbf{P}_{ML}^{-1}, \end{aligned} \tag{22}$$

in which $\mathbf{P}_{11} = \mathbf{P}_{k|k-1}$ and $\mathbf{P}_{22} = \mathbf{P}_{ML}$. It can be also seen that the condition (3) is fulfilled

$$\mathbf{c}_1 + \mathbf{c}_2 = \mathbf{P}_{k|k}(\mathbf{P}_{k|k-1}^{-1} + \mathbf{P}_{ML}^{-1}) = \mathbf{P}_{k|k}\mathbf{P}_{k|k}^{-1} = \mathbf{I}_{n_x}. \tag{23}$$

The posterior covariance matrix $\mathbf{P}_{k|k}$ can be written in terms of Millman's formula (11) as

$$\begin{aligned}\mathbf{P}_{k|k} &= \mathbf{c}_1\mathbf{P}_{11}\mathbf{c}_1^T + \mathbf{c}_2\mathbf{P}_{22}\mathbf{c}_2^T = \\ &= (\mathbf{P}_{11}^{-1} + \mathbf{P}_{22}^{-1})^{-1}(\mathbf{P}_{11}^{-1} + \mathbf{P}_{22}^{-1})(\mathbf{P}_{11}^{-1} + \mathbf{P}_{22}^{-1})^{-1} = \\ &= (\mathbf{P}_{k|k}^{-1} + \mathbf{P}_{ML}^{-1})^{-1} = (\mathbf{P}_{k|k}^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})^{-1}\end{aligned} \tag{24}$$

From relations (20)–(24) it can be seen that the KF measurement update equations (14) and (18) fits the MF (10) and (11).

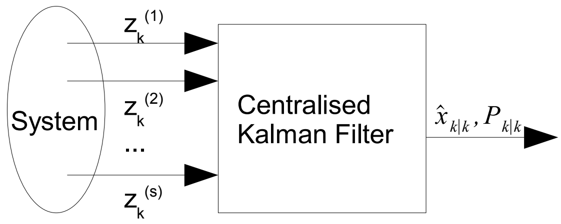

Fig. 1. Centralised fusion

B. Millman's formula in smoothing

In this part the use of the MF in the problem of smoothing is discussed. The difference contrary to filtering is that there are more data available. The filtering estimate with the covariance $\mathbf{P}_{k|k}$ is gained by a standard forward KF. The filtering estimate is smoothed by the data ${}_{k+1}^{l}\mathbf{z}$, where $l > k$ and ${}_{k+1}^{l}\mathbf{z} = [\mathbf{z}_{k+1}, \ldots, \mathbf{z}_l]$. These data are processed by the information filter which runs backward in time with zero initial condition [9]. The backward estimate is equal to

$$\hat{\mathbf{x}}_{k|l} = \mathbf{S}_{k|l}^{-1}\hat{\mathbf{y}}_{k|l}, \tag{25}$$

where $\mathbf{S}_{k|l} = \mathbf{P}_{k|l}^{-1}$ is the precision matrix and $\hat{\mathbf{y}}_{k|l}$ is the information coming from the backward information filter.

To fuse $\hat{\mathbf{x}}_{k|k}$ and $\hat{\mathbf{y}}_{k|l}$ optimally in the mean square error sense, the MF (10) and (11) can be used. The fusion equations are then given as

$$\mathbf{K}_k = \mathbf{P}_{k|k}\mathbf{S}_{k|l}(\mathbf{I}_{n_x} + \mathbf{P}_{k|k}\mathbf{S}_{k|l})^{-1}, \tag{26}$$
$$\mathbf{P}_k = (\mathbf{I}_{n_x} - \mathbf{K}_k)\mathbf{P}_{k|k}, \tag{27}$$
$$\hat{\mathbf{x}}_k = (\mathbf{I}_{n_x} - \mathbf{K}_k)\hat{\mathbf{x}}_{k|k} + \mathbf{P}_k\hat{\mathbf{y}}_{k|l}. \tag{28}$$

The smoothing problem formulation is different from that of filtering problem but the solution is obtained by the same means, the filtering theory and the MF.

V. Generalised Millman's Formula in State Estimation

In this section, the system (12), (13) with multiple sensors, i.e. $S > 1$, is supposed. The fusion of multiple estimates is shown by using the Generalised Millman's formula (6), (7) for some special cases.

A. Optimal centralised fusion

To evaluate multisensor fusion performance, the centralised solution will be introduced. This solution is hypothetical only because it supposes no constraints on the fusion architecture, i.e. all measurements are available to the central processor, see Fig. 1. The multisensor system is rewritten as a one-sensor system by merging the measurement equations (13) to one block-matrix equation

$$\mathbf{z}_k = \mathbf{H}\mathbf{x}_k + \mathbf{v}_k \tag{29}$$

where

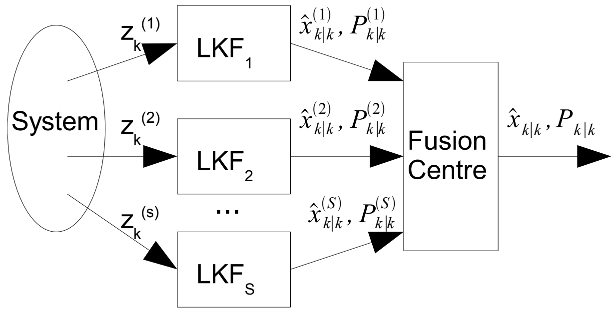

Fig. 2. Multisensor fusion

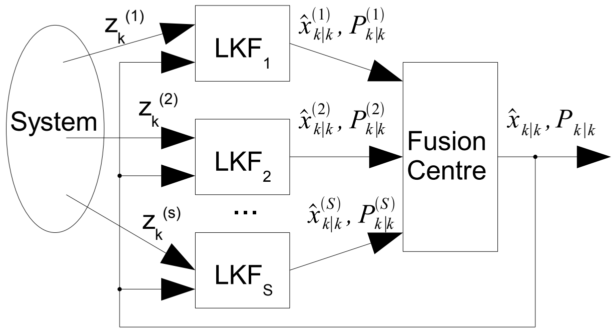

Fig. 3. Hierarchical fusion

$$\mathbf{z}_k = \begin{pmatrix} \mathbf{z}_k^{(1)} \\ \mathbf{z}_k^{(2)} \\ \vdots \\ \mathbf{z}_k^{(S)} \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} \mathbf{H}^{(1)} \\ \mathbf{H}^{(2)} \\ \vdots \\ \mathbf{H}^{(S)} \end{pmatrix}, \quad \mathbf{v}_k = \begin{pmatrix} \mathbf{v}_k^{(1)} \\ \mathbf{v}_k^{(2)} \\ \vdots \\ \mathbf{v}_k^{(S)} \end{pmatrix}, \tag{30}$$

with the measurement noise covariance matrix

$$\mathbf{R} = \begin{pmatrix} \mathbf{R}^{(1)} & 0 & \cdots & O \\ 0 & \mathbf{R}^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{R}^{(S)} \end{pmatrix}. \tag{31}$$

B. Multisensor fusion

The multisensor problem definition assumes that there are multiple estimators which process their corresponding measurements and possibly other state estimates. The simplest fusion architecture consists of one central estimator which fuses the local state estimates. The situation is depicted in Fig. 2 where LKF denotes a local Kalman filter.

It is evident that this problem does not equal to the centralised fusion. Now, the direct application of the MF for the multisensor case will be shown. The system with multiple sensors is considered, i.e. $S > 1$. Each sensor provides an estimate $\hat{\mathbf{x}}^{(j)}_{k|k}$ with associated covariance matrix $\mathbf{P}^{(j)}_{k|k}$, $j = 1, \ldots, S$. If the process noise is sufficiently small, the local estimates can be fused by the GMF as being independent.

For independent estimates, i.e. $P_{ij} = 0$ for $i \neq j$, the solution to the eqs. (6), (7) of the GMF has a simple closed form

$$\mathbf{c}_i = \left(\sum_{j=1}^{N} \mathbf{P}_{jj}^{-1} \right)^{-1} \mathbf{P}_{ii}^{-1}. \tag{32}$$

Applying (32) to (2), (4), the fused estimate $\hat{\mathbf{x}}_{k|k}$ for the multisensor fusion is

$$\hat{\mathbf{x}}_{k|k} = \mathbf{P}_{k|k} \sum_{j=1}^{N} \mathbf{P}^{(j)\,-1}_{k|k} \hat{\mathbf{x}}^{(j)}_{k|k}, \tag{33}$$

where the estimate covariance $\mathbf{P}_{k|k}$ is given by

$$\mathbf{P}^{-1}_{k|k} = \sum_{j=1}^{N} \mathbf{P}^{(j)\,-1}_{k|k}. \tag{34}$$

Note that the assumption of independence is not correct if the process noise is non-zero, i.e. $\mathbf{Q} > \mathbf{0}$. In such case, the estimates are dependent despite the measurements are independent.

The exact solution to the eqs. (6), (7) requires computation of cross-covariances $\mathbf{P}^{(ij)}_{k|k}$. For the system (12), (13), these covariances are given by [7]

$$\begin{aligned} \mathbf{P}^{(ij)}_{k|k} = & \left[\mathbf{I}_{n_x} - \mathbf{K}^{(i)}_k \mathbf{H}^{(i)} \right] \times \left[\mathbf{F} \mathbf{P}^{(ij)}_{k-1|k-1} \mathbf{F}^{\mathrm{T}} + \mathbf{G} \mathbf{Q} \mathbf{G}^{\mathrm{T}} \right] \\ & \times \left[\mathbf{I}_{n_x} - \mathbf{K}^{(j)}_k \mathbf{H}^{(j)} \right]^{\mathrm{T}} \end{aligned} \tag{35}$$

where $\mathbf{P}_{0|0} = \mathbf{P}_0$ and $\mathbf{K}^{(i)}_k$ is the Kalman gain of the i-th local Kalman filter.

C. Hierarchical fusion

The hierarchical fusion is a generalisation of the previous multisensor fusion case. A feedback from the fusion center (FC) to all local estimators is introduced. The problem is illustrated in Fig. 3.

To cope with the problem of the dependency of the local estimates, the FC extracts the measurement information from the estimates available to the FC. The so called equivalent measurements are gained by a reverse use of the MF in (19). Then the independent information coming from each sensor is fused with the prior estimate $\hat{\mathbf{x}}_{k|k-1}$, $\mathbf{P}_{k|k-1}$ by using the GMF closed form solution (32) that results to

$$\begin{aligned} \mathbf{P}^{-1}_{k|k} \hat{\mathbf{x}}_{k|k} = & \mathbf{P}^{-1}_{k|k-1} \hat{\mathbf{x}}_{k|k-1} + \\ & + \sum_{j=1}^{S} \left\{ \mathbf{P}^{(j)\,-1}_{k|k} \hat{\mathbf{x}}^{(j)}_{k|k} - \mathbf{P}^{(j)\,-1}_{k|k-1} \hat{\mathbf{x}}^{(j)}_{k|k-1} \right\}, \end{aligned} \tag{36}$$

$$\mathbf{P}^{-1}_{k|k} = \mathbf{P}^{-1}_{k|k-1} + \sum_{j=1}^{S} \left\{ \mathbf{P}^{(j)\,-1}_{k|k} - \mathbf{P}^{(j)\,-1}_{k|k-1} \right\}. \tag{37}$$

By using the feedback to restart the local filters, the one-step predictions are replaced by the fused estimate prediction

$$\hat{\mathbf{x}}^{(j)}_{k+1|k} = \hat{\mathbf{x}}_{k+1|k}\,, \quad \mathbf{P}^{(j)}_{k+1|k} = \mathbf{P}_{k+1|k} \tag{38}$$

where the one-step central prediction is

$$\hat{\mathbf{x}}_{k+1|k} = \mathbf{F}\cdot\hat{\mathbf{x}}_{k|k}\,, \tag{39}$$

$$\mathbf{P}_{k+1|k} = \mathbf{F}\cdot\mathbf{P}_{k|k}\cdot\mathbf{F}^T + \mathbf{G}\cdot\mathbf{Q}\cdot\mathbf{G}^T. \tag{40}$$

The local filters are standard Kalman filters. At (36), respectively at (37), the equivalent measurements are represented by the differences

$$\mathbf{P}^{(j)\,-1}_{k|k}\hat{\mathbf{x}}^{(j)}_{k|k} - \mathbf{P}^{(j)\;\;-1}_{k|k-1}\hat{\mathbf{x}}^{(j)}_{k|k-1}$$

with the equivalent measurement covariance matrices

$$\mathbf{P}^{(j)\,-1}_{k|k} - \mathbf{P}^{(j)\;\;-1}_{k|k-1}.$$

Note that, the fused estimates (36) and (37) are equivalent to those obtained by the centralised KF [4], [10]. This filter is also called distributed Kalman filter since it distributes the computation load of the centralised Kalman filter to more filters.

VI. Numerical Illustration

In this section a numerical example representing simple tracking problem is given. The example illustrates estimation performance of multi-sensor fusion techniques which were introduced in Section V.

Consider a state equation in the form [10]

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + \mathbf{w}_k, \tag{41}$$

where $x_{1,k}$ is position of an object, $x_{2,k}$ its velocity, and T_s is the sampling period. The state noise $\mathbf{w}_k$ is zero mean with the covariance matrix

$$\mathbf{Q}_k = \begin{bmatrix} \frac{1}{3}T_s^3 & \frac{1}{2}T_s^2 \\ \frac{1}{2}T_s^2 & T_s \end{bmatrix} N_0 \tag{42}$$

with the power spectral density of the noise $N_0 = 0.16T_s$, $\forall k$.

The position of the object in Cartesian coordinates is measured by three sensors, i.e. $S = 3$,

$$\mathbf{z}^{(j)}_k = [1\ 0]\mathbf{x}_k + v^{(j)}_k, \tag{43}$$

where $v^{(j)}_k$ is the measurement noise with zero mean and variance $R = 1$ for $j = 1,2,3$ and $\forall k$. The particular measurement noises are independent to each other.

The initial condition of the object and of the filters is assumed as $p(\mathbf{x}_0) = p(\mathbf{x}_0|\mathbf{z}^{-1}) = \mathcal{N}\{\mathbf{x}_0 : [100\ \ -1]^T, \mathrm{diag}([1\ \ 1])\}$, where $\mathrm{diag}(\mathbf{y})$ represents diagonal matrix with the vector $\mathbf{y}$ on the main diagonal. The system were run for $k = 0,1,\ldots,K$, $K = 20$, time instants.

Five different estimators were compared, namely

- the *centralised (optimal) KF* (CKF) - all measurements are processed by one filter and thus no estimates are communicated (Section V-A),
- the *hierarchical KF* (HKF) - each measurement is processed by a single KF and the resulting estimates are fused according to relations (36)–(37) and the global (fused) estimate is provided to all particular (local) KF's (Section V-C)
- the *multi-sensor case* (MSC) - each measurement is processed by a single KF and the resulting estimates are fused according to relations (33)–(34) but the global estimate is not provided to the local filters (Section V-B),
- the *multi-sensor case with GMF used in the fusion centre* (MSC-GMF) - same structure as the MSC, but the local estimates are fused according to the generalised Millman's formula, which was introduced in Section 2 (exact GMF relations valid for S KF's can be found e.g. in [7]), and
- the *single KF* (SKF), which process measurement from one sensor only.

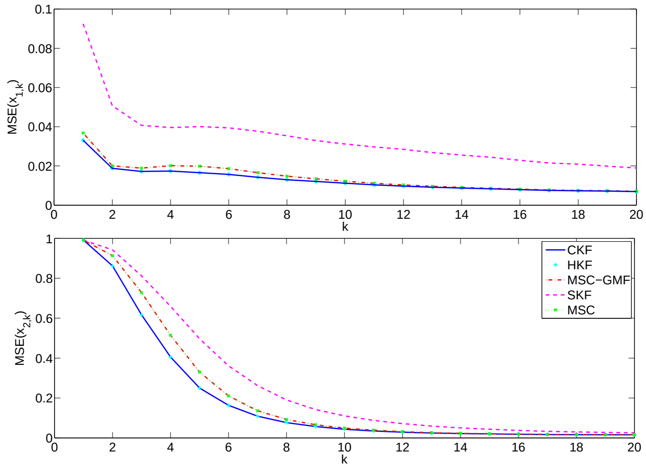

Fig. 4. Time behaviour of the mean square error

estimator	CKF	HKF	MSC-GMF	MSC	SKF
MSE	0.1010	0.1010	0.1134	0.1134	0.1528

TABLE I

Overall mean square error

The performance of the considered filters has been compared using the mean square error (MSE), trace of the filtering covariance matrix $\mathbf{P}_{k|k}$ and the Mahalanobis distance.

The MSE for both states, which is given by

$$MSE(x_{i,k}) = \frac{\sum_{m=1}^{M}(x_{i,k,m} - \hat{x}_{i,k|k,m})^2}{M}, \tag{44}$$

where $x_{i,k,m}$ is the i-th component of the true state at time k in the m-th Monte-Carlo simulation and $\hat{x}_{i,k|k,m}$ its (global) filtering estimate, $i = 1,2$, are shown in Fig. 4 for $M = 10^4$. Fig. 5 shows the differences in the MSE between the optimal CKF and the other filters.

The overall mean square error

$$MSE = \frac{\sum_{i=1}^{2}\sum_{k=0}^{K} MSE(x_{i,k})}{2(K+1)} \tag{45}$$

are given in Table I.

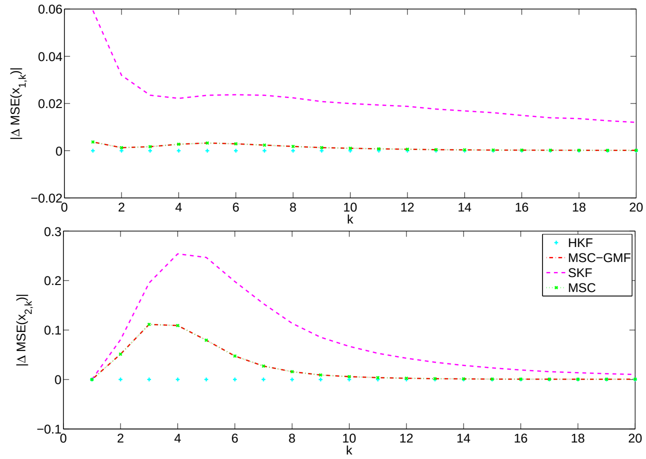

Fig. 5. Differences in the MSE between the CKF and the other filters

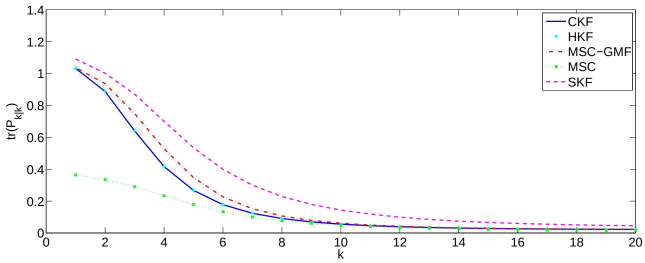

Fig. 6. Traces of the filtering covariance matrices

The traces of the filtering covariance matrices are shown in Fig. 6.

The Mahalanobis distance is defined by

$$D_{\mathscr{M}}(\mathbf{x}_k, \hat{\mathbf{x}}_{k|k}) = \sqrt{(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k})^{\mathrm{T}} \mathbf{P}_{k|k}^{-1} (\mathbf{x}_k - \hat{\mathbf{x}}_{k|k})}. \tag{46}$$

The overall Mahalanobis distance computed according to

$$D_{\mathscr{M}} = \frac{\sum_{k=0}^{K} \sum_{m=1}^{M} D_{\mathscr{M}}(\mathbf{x}_{k,m}, \hat{\mathbf{x}}_{k|k,m})}{(K+1)M} \tag{47}$$

is summarised in Tab. II.

In this example, the CKF represents the optimal mean square error estimator. From Tab. I and Figs. 4, 5, it can be seen that the HKF provides also the optimal estimates, what is in accord with theoretical conclusions given in Section V-C. However, the HKF is more preferable for systems with many sensors, where the computational demands of the CKF need to be distributed to local filters. The further estimators representing multi-sensor fusion, i.e. the MSC and the MSC-GMF, gives suboptimal estimates only. The reason is twofold; the global estimates are not provided to local KF's, i.e. the local estimators do not comprise information from other sensors, and the relations for global estimates computation are not optimal in the mean square error sense. On the other hand, advantage of multi-sensor fusion can be found in reduction of required bandwidth for communication among all estimators. For completeness, the estimation results of the SKF was shown to emphasise the effect of multiple measurements.

estimator	CKF	HKF	MSC-GMF	MSC	SKF
$D_{\mathscr{M}}$	1.2512	1.2512	1.2536	1.5204	1.2550

TABLE II
OVERALL MAHALANOBIS DISTANCE

The mean square error comparison consider the mean estimates only. The trace of the covariance matrix represents the uncertainty of the estimate. The trace also allows to show the consequences of ignoring the dependency of the estimates. The time evolution of the traces of the filtering covariance matrices is plotted in Fig. 6. The MSC filter underestimates the covariance matrices that causes the MSC estimates are over-confident. The Mahalanobis distance evaluates the estimated mean and covariance matrix simultaneously. The over-confidence can be seen also from Tab. II, the overall Mahalanobis distance of the MSC estimates is far greater than the other ones.

VII. CONCLUDING REMARKS

The paper dealt with state estimation from the data fusion point of view. The standard relations for filtering, smoothing, multisensor and hierarchical fusion were introduced by using the generalised Millman's formula. The hierarchical and multisensor fusion were compared with the optimal centralised Kalman filter by means of a numerical example. The hierarchical fusion gives results which are equivalent to the centralised fusion, the application of the generalised Millman's formula gives suboptimal estimates only. The approximation of the generalised Millman's formula by neglecting cross-correlations leads to over-confident estimates.

VIII. ACKNOWLEDGEMENTS

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