# Analytical and LMI based design for the Acrobot traking with aplication to robot walking

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*Abstract*— **This paper aims to compare the performance of various techniques for the stabilization of the error dynamics of the Acrobot's walking like reference trajectory. Both the walking reference planning and the tracking feedback design are based on the Acrobot's model partial exact feedback linearization of order 3. Namely, such an exact system transformation leads to an almost linear system where error dynamics along trajectory to be tracked is a 4 dimensional linear time varying system having 3 time varying entries only, the remaining entries are either zero or equal to one.**

## I. INTRODUCTION

Underactuated mechanical systems are those having less actuators than degrees of freedom. Efficient control of underactuated mechanical systems constitutes one of the most challenging problems of recent decades, see [17], [7] and references therein. Walking like mechanical chain systems have typically one non-actuated joint which is at the support pivot point of the walking like mechanism. Reliable and economic walking is the typical example of the related studies among both control and robotic community. The corresponding problems are basically related with a) walking like trajectory planning, and b) design of the feedback ensuring exponential tracking of such a trajectory. One of the simplest underactuated mechanical systems is the Acrobot, depicted on Figure 1, sometimes called also as the biped. Despite being a seemingly simple system, the Acrobot comprises many important features of underactuated walking robots having degree of underactuation equal to one. Recently, numerous papers have addressed the stabilization of its inverted position extending its domain of attraction [3], [14],[8],[5], [18],[16], or even stable walking-like movement [6], [2], [1]. Despite its simplicity, the Acrobot comprises all typical problems related to control of the underactuated walking like mechanical systems. As a consequence, the effective control of the acrobot is an important step on the route to underactuated walking, see [6], [2], [1], [5], [18] for more detailed arguments.

This paper aims to further extend the results obtained in [6], [2], [1] regarding exponential tracking of the pre-selected walking like target trajectory. This design is based on the partial exact feedback linearization of the order 3. Namely, such an exact system transformation leads to an almost linear system where error dynamics along trajectory to be tracked is a 4 dimensional linear time varying system where the corresponding right hand side matrix has 3 time varying entries only, the remaining entries are either zero or equal to one.

The rest of the paper is organized as follows. The next section briefly presents the model of the Acrobot together with the main theoretical pre-requisites necessary for further numerical analysis. Section 3 describes the main result of this paper. Simulations of Acrobot walking are presented in Section 4. The final section draws briefly some conclusions and discusses some open future research outlooks toward efficient underactuated walking.

## II. THE MODEL OF THE ACROBOT

The acrobot depicted on Figure 1 is a special case of $n$-link chain with $n-1$ actuators attached by one of its ends

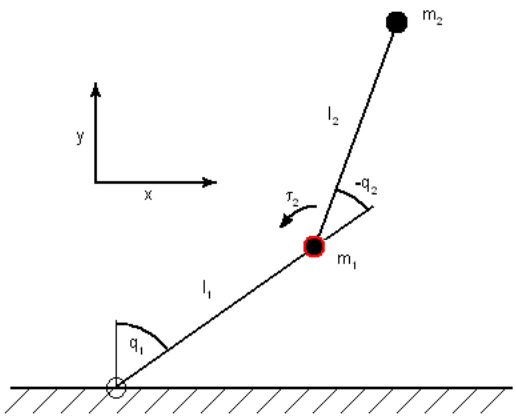


Fig. 1. Acrobot.

to a pivot point through an unactuated rotary joint. Such a system can be modelled by usual Lagrangian approach [9]. The corresponding Lagrangian is as follows

$$\mathcal{L}(q,\dot{q}) = K - V = \frac{1}{2}\dot{q}^T D(q)\dot{q} - V(q) \qquad (1)$$

where $q$ denotes a $n$-dimensional configuration vector on the configuration manifold $Q$ and $D(q)$ is the inertia matrix, $K$ is the kinetic energy and $V$ is the potential energy of the

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system. The resulting Euler-Lagrange equation is

$$\begin{bmatrix} \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_1} - \frac{\partial\mathcal{L}}{\partial q_1} \\ \vdots \\ \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_n} - \frac{\partial\mathcal{L}}{\partial q_n} \end{bmatrix} = u = \begin{bmatrix} 0 \\ \tau_2 \\ \vdots \\ \tau_n \end{bmatrix}, \qquad (2)$$

where $u$ stands for vector of external controlled forces. The system (2) is the so-called **underactuated** mechanical system having the degree of the underactuation equal to one, [15]. Moreover, the underactuated angle is at the pivot point. Equation (2) leads to a dynamic equation in the form

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u \qquad (3)$$

where $D(q)$ is the inertia matrix, $C(q,\dot{q})$ contains Coriolis and centrifugal terms, $G(q)$ contains gravity terms and $u$ stands for vector of external forces.

For the Acrobot, these computations lead to a second-order nonholonomic constraint and a kinetic symmetry, i.e. the inertia matrix depends only on the second variable $q_2$

$$D(q) = \left[\begin{array}{l|r} \theta_1 + \theta_2 + 2\theta_3 \cos q_2 & \theta_2 + \theta_3 \cos q_2 \\ \theta_2 + \theta_3 \cos q_2 & \theta_2 \end{array}\right], \qquad (4)$$

$$C(q,\dot{q}) = \begin{bmatrix} -\theta_3 \sin q_2 \dot{q}_2 & -(\dot{q}_2 + \dot{q}_1)\theta_3 \sin q_2 \\ \theta_3 \sin q_2 \dot{q}_1 & 0 \end{bmatrix}, \qquad (5)$$

$$G(q) = \begin{bmatrix} -\theta_4 g \sin q_1 - \theta_5 g \sin(q_1 + q_2) \\ -\theta_5 g \sin(q_1 + q_2) \end{bmatrix}, \qquad (6)$$

where the 2-dimensional configuration vector $(q_1, q_2)$ consists of angles defined on Figure 1 and

$$\begin{aligned} \theta_1 = (m_1 + m_2)l_1^2 + I_1,\ \theta_2 = m_2 l_2^2 + I_2, \\ \theta_3 = m_2 l_1 l_2,\ \theta_4 = (m_1 + m_2)l_1,\ \theta_5 = m_2 l_2. \end{aligned} \qquad (7)$$

The **partial exact feedback linearization** method is based on a system transformation into a new system of coordinates that displays linear dependence between some auxiliary output and new (virtual) input [13]. From the theoretical point of view, the $n$-degrees of freedom mechanical system dynamics is described by $2n$-dimensional state space. Static state feedback linearization generated by the suitable output function having the relative degree $r$ yields a linear subsystem of dimension $r$. In other words, the maximal feedback linearization problem consists in finding a function with maximal relative degree. In [12] it was shown that if the generalized momentum conjugate to the cyclic variable is not conserved (as it is the case of Acrobot) then there exists a set of outputs that defines a one-dimensional exponentially stable zero dynamics. That means that it is possible to find a function $\overline{y}(q,\dot{q})$ with relative degree 3 that transforms the original system 4 dimensional system (33) by a local coordinate transformation $z = T(q,\dot{q})$ into the new input/output linear system having 3 dimensional state plus unobservable nonlinear dynamics of dimension 1.

In the case of the Acrobot there are two independent functions with relative degree 3 transforming the system into the desired partial linearized form with one dimensional zero dynamics,[1] namely

$$\begin{aligned} \sigma &= \frac{\partial\mathcal{L}}{\partial\dot{q}_1} = (\theta_1 + \theta_2 + 2\theta_3 \cos q_2)\dot{q}_1 + \\ &\qquad (\theta_2 + \theta_3 \cos q_2)\dot{q}_2, \end{aligned} \qquad (8)$$

$$\begin{aligned} p &= q_1 + \frac{q_2}{2} + \frac{2\theta_2 - \theta_1 - \theta_2}{\sqrt{(\theta_1+\theta_2)^2 - 4\theta_3^2}} \arctan \\ &\qquad \left(\sqrt{\frac{\theta_1 + \theta_2 - 2\theta_3}{\theta_1 + \theta_2 + 2\theta_3}} \tan\frac{q_2}{2}\right). \end{aligned} \qquad (9)$$

The zero dynamics is used to investigate the internal stability when the corresponding output is forced to zero. For the most simple cases $\overline{y} = Cp$ or $\overline{y} = C\sigma$ the resulting zero dynamics is only critically stable. However, considering the output function $\overline{y} = C_1 p(q) + C_2 \sigma(q,\dot{q})$ one gets the following zero dynamics $\dot{p} + C_1[C_2 d_{11}(q_2)]^{-1} p = 0$ which is asymptotically stable whenever $C_1/C_2$ is positive, $d_{11}(q_2)$ is the corresponding part of the inertia matrix $D$ in (33). Unfortunately, the corresponding transformations have a complex set of singularities, unless $C_1$ is very small, which is not suitable for practical purposes.

In [6] it was shown that using the set of functions with maximal relative degree, the following transformation

$$T:\ \ \xi_1 = p, \xi_2 = \sigma, \xi_3 = \dot{\sigma}, \xi_4 = \ddot{\sigma} \qquad (10)$$

can be defined. Notice, that by (8,9) and some straightforward but laborious computations the following relation holds

$$\dot{p} \ = \ d_{11}(q_2)^{-1}\sigma, \qquad (11)$$

where $d_{11}(q_2) = (\theta_1 + \theta_2 + 2\theta_3 \cos q_2)$ is the corresponding element of the inertia matrix $D$ in (33). Applying (10), (11) to (33) we obtain the Acrobot's dynamics in partial exact linearized form

$$\begin{aligned} \dot{\xi}_1 = d_{11}(q_2)^{-1}\xi_2,\ \dot{\xi}_2 = \xi_3,\ \dot{\xi}_3 = \xi_4, \\ \dot{\xi}_4 = \alpha(q,\dot{q})\tau_2 + \beta(q,\dot{q}) = w \end{aligned} \qquad (12)$$

with the new coordinates $\xi$ and the input $w$ being well defined wherever $\alpha(q,\dot{q})^{-1} \neq 0$.

Assume that an open loop control $w^r(t)$, generating a suitable reference trajectory, is given in partial exact linearized coordinates (12). In other words, our task is to track the following reference system

$$\dot{\xi}_1^r = d_{11}^{-1}(q_2^r)\xi_2^r,\ \dot{\xi}_2^r = \xi_3^r,\ \dot{\xi}_3^r = \xi_4^r,\ \dot{\xi}_4^r = w^r. \qquad (13)$$

Denoting $e := \xi - \xi^r$ and subtracting (13) from (12) one obtains

$$\dot{e}_1 = d_{11}^{-1}(\phi_2(\xi_1,\xi_3))\xi_2 - d_{11}^{-1}(\phi_2(\xi_1^r,\xi_3^r))\xi_2^r,$$

$$\dot{e}_2 = e_3,\quad \dot{e}_3 = e_4,\quad \dot{e}_4 = w - w^r.$$

[1] Actually, by (2) $\dot{\sigma} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_1} = \frac{\partial\mathcal{L}}{\partial q_1}$ and therefore by (1) $\dot{\sigma} = -\frac{\partial V(q)}{\partial q_1}$ as $D(q) \equiv D(q_2)$ by (4). In other words, $\dot{\sigma}$ has relative degree 2, i.e. $\sigma$ has the relative degree 3. Moreover, by the straightforward differentiation it holds $\dot{p} = d_{11}(q_2)^{-1}\sigma$, i.e. $\dot{p}$ has relative degree 2, i.e. $p$ should have relative degree 3 as well.

Straightforward computations based on the Taylor expansions give

$$\begin{aligned} \dot{e}_1 &= \mu_2(t)e_2 + \mu_1(t)e_1 + \mu_3(t)e_3 + o(e) \qquad (14) \\ \dot{e}_2 &= e_3, \quad \dot{e}_3 = e_4, \quad \dot{e}_4 = w - w^r, \qquad (15) \end{aligned}$$

where $\mu_1(t), \mu_2(t), \mu_3(t)$ are known smooth time functions:

$$\mu_1(t) = \xi_2^r(t)\frac{\partial[d_{11}^{-1}]}{\partial q_2}\frac{\partial \phi_2}{\partial \xi_1}(q_2^r(t)), \qquad (16)$$

$$\mu_2(t) = d_{11}^{-1}(q_2^r(t)), \qquad (17)$$

$$\mu_3(t) = \xi_2^r(t)\frac{\partial[d_{11}^{-1}]}{\partial q_2}\frac{\partial \phi_2}{\partial \xi_3}(q_2^r(t)), \qquad (18)$$

$$q_2^r(t) = \phi_2(\xi_1^r(t), \xi_3^r(t)), \quad q_2^r \in [0, 2\pi). \qquad (19)$$

In [6] it was shown that

$$|\mu_1(t)| \le 2\theta_3 a_{max}^2(\theta_4 + \theta_5)\frac{\mathcal{R}}{\mathcal{B}} \qquad (20)$$

$$|\mu_3(t)| \le 2\theta_3 a_{max}^2 \frac{\mathcal{R}}{\mathcal{B}}, \quad 0 < a_{min} \le \mu_2(t) \le a_{max}. \qquad (21)$$

Let us repeat the main results in [2] and [1]

### A. LMI based stabilization of the error dynamics

In [2] it was shown that for reference trajectory tracking one has to solve the following stabilization problem. Consider the open-loop continuous time-varying linear system

$$\dot{e} = A(t)e + Bu, \qquad (22)$$

where

$$A(t) = \begin{pmatrix} \mu_1(t) & \mu_2(t) & \mu_3(t) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The tracking problem consists in finding the state-feedback controller

$$u = Ke, \quad K = \begin{pmatrix} K_1 & K_2 & K_3 & K_4 \end{pmatrix}, \qquad (23)$$

producing the following closed-loop system

$$\dot{e} = (A + BK)\, e = \begin{pmatrix} \mu_1(t) & \mu_2(t) & \mu_3(t) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ K_1 & K_2 & K_3 & K_4 \end{pmatrix} e, \qquad (24)$$

where bounds for $\mu(t) = (\mu_1(t), \mu_2(t), \mu_3(t))$ are given by (20)-(21).

Despite entries of $\mu(t)$ are **known** functions, the appealing idea is to treat them as **unknown disturbances** satisfying the above mentioned given constraints. If constraints are tight enough, one can think about solving quadratic stability conditions and design a unique feedback stabilizing such an "uncertain" system. Obviously, such a feedback would be at the same time solving our tracking problem.

Consider the well-known Lyapunov inequality to be solved for all values of $\mu(t)$ by finding a suitable symmetric positive definite matrix $S$ and a vector $K$:

$$(A(\mu) + BK)^{\mathrm{T}} S + S(A(\mu) + BK) \preceq 0, \qquad (25)$$

$$S = S^{\mathrm{T}} \succ 0. \qquad (26)$$

Such a problem is in fact bilinear with respect to the unknowns. Denoting

$$Q = S^{-1}, Y = KS^{-1} \qquad (27)$$

we derive the following LMI condition for quadratically stabilizing feedback design:

$$QA^{\mathrm{T}}(\mu) + A(\mu)Q + Y^{\mathrm{T}}B^{\mathrm{T}} + BY \preceq 0. \qquad (28)$$

Notice that the pair $(A(\mu), B)$ is controllable if and only if

$$\mu_1\mu_3 + \mu_2 \neq 0. \qquad (29)$$

Obviously, if the set of possible values of $\mu$ contains, or stays close to, the singular set given by (29), LMI (28) becomes infeasible, or almost infeasible.

### B. Analytical design of the exponential tracking

In [1] it was shown that using the precise knowledge of the function $\mu_3$ and knowledge of the ranges of values of $\mu_{1,2,3}(t), \dot{\mu}_3(t)$ then the system (13) could be transformed into following linear time varying system

$$\begin{aligned} \dot{e}_1 &= \mu_2(t)e_2 + \mu_1(t)e_1 + \mu_3(t)e_3 \\ \dot{e}_2 &= e_3, \\ \dot{e}_3 &= e_4, \\ \dot{e}_4 &= \Theta^3\widetilde{K}_1 e_1 \\ &+ \Theta^3[\widetilde{K}_2 - \widetilde{K}_1\mu_3(t)]e_2 + \Theta^2\widetilde{K}_3 e_3 + \Theta\widetilde{K}_4 e_4, \end{aligned} \qquad (30)$$

where time functions $\mu_{1,2,3}(t)$ are such that $\forall t \ge 0$ it holds:

$$\begin{aligned} |\mu_1| &\le M^1, \\ 0 < M_2 \le \mu_2(t) &+ \mu_1(t)\mu_3(t) - \dot{\mu}_3(t) \le M^2, \end{aligned} \qquad (31)$$

for some suitable real constants $M^1, M_2, M^2$. Assume that $\widetilde{K}_{2,3,4}$ are such that the polynomial $\lambda^3 + \widetilde{K}_4\lambda^2 + \widetilde{K}_3\lambda + \widetilde{K}_2$ is Hurwitz and, moreover,

$$M^1 - M_2\frac{\widetilde{K}_1}{\widetilde{K}_2} < 0. \qquad (32)$$

In [1] there is the proof that exists a sufficiently large $\Theta > 0$ such that the system (30) is globally exponentially stable.

## III. Impact model

An impact occurs when the swing leg touches the walking surface. The impact between the swing leg and the ground is modeled as a contact between two rigid bodies. There are many different ways how impact can be modeled.

For development of impact rules, the dynamic model (33) has to be enlarged by reaction force effects. By adding Cartesian coordinates $(p^h{}_H, p^v{}_H)$ to the hip, the following extended model is obtained

$$D_e(q_e)\ddot{q}_e + C_e(q_e, \dot{q}_e)\dot{q}_e + G_e(q_e) = B_e u + \delta F_{ext} \qquad (33)$$

where $q_e = (q_1, q_2, p^h{}_H, p^v{}_H)$ and $\delta F_{ext}$ represents the vector of external forces acting on the robot at the contact point.

If there is no slip or rebound, the positions q do not change during the impact $q^+ = q^-$. Suppose the stance leg tip is in contact with the ground and not slipping, then the extended coordinates $q_e$ and their velocities $\dot{q}_e$ are related to $q$ and $\dot{q}_e$ by

$$q_e = \Upsilon(q), \;\; \dot{q}_e = \frac{\partial \Upsilon(q)}{\partial q}\dot{(q)} \tag{34}$$

where$\Upsilon = (q', p^h{}_H, p^v{}_H)'$. The impact model of [10] is used under the following assumptions that imply that the total angular momentum is conserved:

H1 The contact of the swing leg with the ground results in no rebound and no slipping of the swing leg.
H2 At the moment of impact, the stance leg lifts from the ground without interaction.
H3 The impact is instantaneous.
H4 The external forces during the impact can be represented by impulses.
H5 The impulsive forces may result in an instantaneous change in the velocities, but there is no instantaneous change in the configuration.
H6 The actuators cannot generate impulses and, hence, can be ignored during impact.

Following an identical development as in [11], the expression relating the velocity of the robot just before to just after the impact may be written as

$$\Pi^{-1}(q_e^-)\begin{bmatrix} \dot{q}_e^+ \\ F_2^T \\ F_2^N \end{bmatrix} = \begin{bmatrix} D_e(q_e^-)\dot{q}_e^+ \\ 0 \end{bmatrix}, \tag{35}$$

where

$$\Pi(q_e^-) = \begin{bmatrix} D_e(q_e) & -\left(\frac{\partial E(q_e)}{\partial q_e}\right)' \\ \frac{\partial E(q_e)}{\partial q_e} & 0 \end{bmatrix}^{-1} \tag{36}$$

and $E(q_e) = (p^h{}_H, p^v{}_H)$ stands for the fixed coordinates of the robot in the Cartesian frame and $(F_2^T, F_2^N)$ are the integrals of the tangential and normal components of the impulsive forces.

Solving (35) yields

$$\begin{bmatrix} \dot{q}_e^+ \\ F_2^T \\ F_2^N \end{bmatrix} = \Pi(q_e^-)\begin{bmatrix} D_e(q_e^-)\dot{q}_e^+ \\ 0 \end{bmatrix}, \tag{37}$$

By partitioning $\Pi(q_e^-)$ the map from $\dot{q}_e^-$ to $\dot{q}_e^+$ is obtained as

$$\begin{aligned} \dot{q}_e^+ &= \Pi_{11}(q_e^-)D_e(q_e^-)\dot{q}_e^- \\ \begin{bmatrix} F_2^T \\ F_2^N \end{bmatrix} &= \Pi_{11}(q_e^-)D_e(q_e^-)\dot{q}_e^- \end{aligned} \tag{38}$$

The combination of (34) with (37) and (38) results in an expression for the velocities of the robot just after impact and the integral of the impulsive forces. At impact, it is assumed that the swing leg becomes the new stance leg and the coordinates must be relabeled.

## IV. SIMULATIONS

To demostrate the Impact we use the same walking like trajectory[2] as in [6], [2], [1]. In means that the first parts of Fig. 2 and Fig. 3 are same as in [2]. The same state-feedback matrix $K = 10^4(-1.9087 - 1.2097 - 0.1781 - 0.0090)$ is used and saturation limit in the range $\pm 10\,\mathrm{Nm}$ was used too. The first parts of Fig. 4 and Fig. 5 is same as in [1]. The same gains $(\widetilde{K}_1, \widetilde{K}_2, \widetilde{K}_3, \widetilde{K}_4) = -(1.5\times 6, 6, 12, 8)$ and the same "amplifying" parameter $\Theta = 20$ was used. The initial positions errors were zero while velocities errors were about 20%.

When the swing led touches the ground occurs impact. At impact the coordinates must be relabeled because the swing led becomes the new stance leg.

The second part of each Figure is produced the same way as the first part. Only the initial conditions are different. In both cases the error in initial conditions is less than at the beginning of the first step.

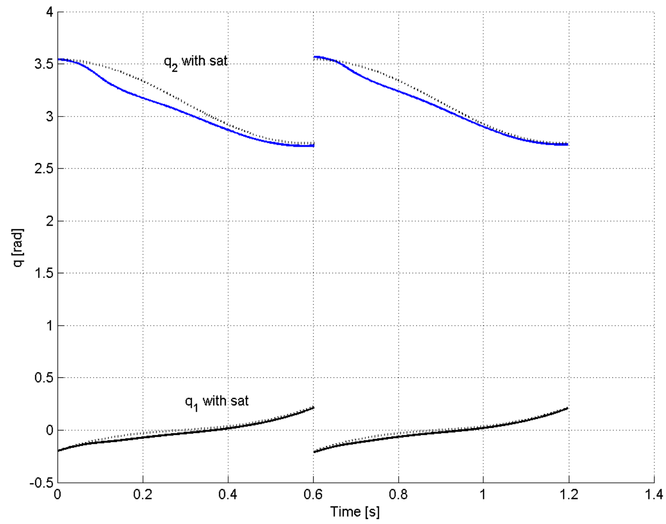


Fig. 2. Angular positions $q_1$, $q_2$ with saturation and references (dotted line). LMI based design.

## V. CONCLUSIONS AND FUTURE WORKS

Using the impact we extend results obtained LMI based design or analytical based design. In both cases the error in initial conditions is less than at the beginning of the first step. Ongoing research is to propose the reference trajectory so that the initial conditions of new step after impact are equal to initial conditions of the reference step.

[2]This trajectory is the so-called pseudo-passive trajectory. Namely, pseudopassive trajectory is the one for which $w^r \equiv 0$, i.e. there is no input action in the exact feedback linearized coordinates. The word "pseudo" expresses the fact that real torque is not zero, but $\tau_2 = -w\alpha(q,\dot{q})/\beta(q,\dot{q})$, due to the linearizing relation between real torque $\tau_2$ and the virtual input $w$ in the partial exact linearized form.

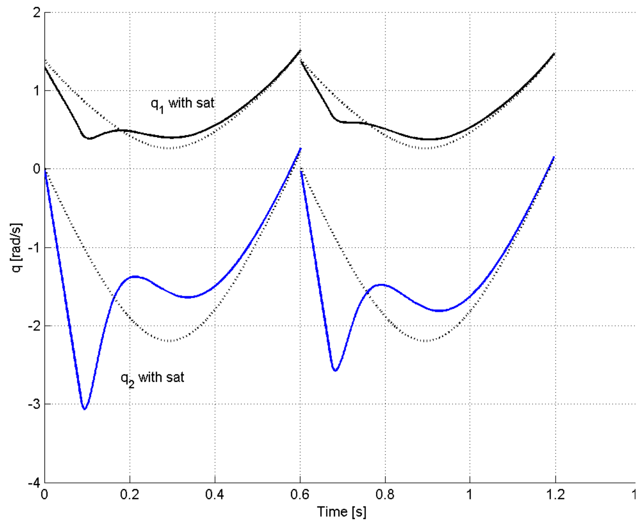


Fig. 3. Angular velocities $q_1$, $q_2$ with saturation and references (dotted line). LMI based design.

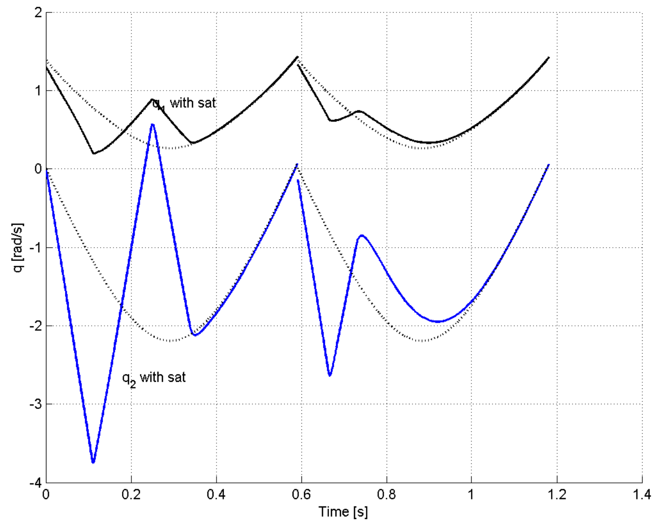


Fig. 5. Angular velocities $q_1$, $q_2$ with saturation and references (dotted line). Analytical based design.

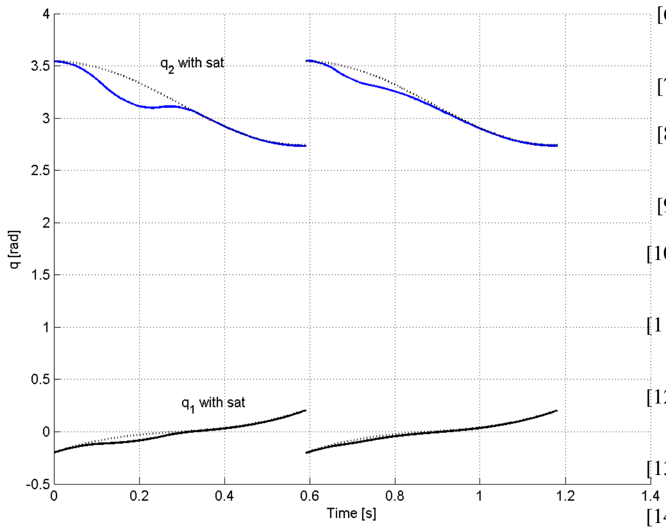


Fig. 4. Angular positions $q_1$, $q_2$ with saturation and references (dotted line). Analytical based design.

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