SIAM J. OPTIM.
Vol. 20, No. 1, pp. 416–444

SHAPE OPTIMIZATION IN THREE-DIMENSIONAL CONTACT PROBLEMS WITH COULOMB FRICTION*

P. BEREMLIJSKI†, J. HASLINGER‡, M. KOČVARA§, R. KUČERA¶, AND J.V. OUTRATA‖

Abstract. We study the discretized problem of the shape optimization of three-dimensional (3D) elastic bodies in unilateral contact. The aim is to extend existing results to the case of contact problems obeying the Coulomb friction law. Mathematical modeling of the Coulomb friction problem leads to an implicit variational inequality. It is shown that for small coefficients of friction the discretized problem with Coulomb friction has a unique solution and that this solution is Lipschitzian as a function of a control variable describing the shape of the elastic body. The 2D case of this problem was studied by the authors in [P. Beremlijski, J. Haslinger, M. Kočvara, and J. V. Outrata, *SIAM J. Optim.*, 13 (2002), pp. 561–587]; there we used the so-called implicit programming approach combined with the generalized differential calculus of Clarke. The extension of this technique to the 3D situation is by no means straightforward. The main source of difficulties is the nonpolyhedral character of the second-order (Lorentz) cone, arising in the 3D model. To facilitate the computation of the subgradient information, needed in the used numerical method, we exploit the substantially richer generalized differential calculus of Mordukhovich. Numerical examples illustrate the efficiency and reliability of the suggested approach.

Key words. shape optimization, contact problems, Coulomb friction, mathematical programs with equilibrium constraints

AMS subject classifications. 49Q10, 74M10, 74S05

DOI. 10.1137/080714427

1. Introduction and preliminaries.

Contact shape optimization is a special branch of structural optimization whose goal is to find shapes of deformable bodies which are in mutual contact. A typical problem in many applications is to find shapes guaranteeing a priori given stress distributions on parts in contact [1]. A specific feature of contact shape optimization is its nonsmooth character due to the fact that the respective state mapping is given by various types of variational inequalities. For contact problems without friction or with the so-called given friction (see [9]), whose mathematical models lead to variational inequalities of the first and the second kind, sensitivity analysis was done in [26] for continuous models and in [10] for discretized models. Assuming a more realistic Coulomb law of friction, the situation becomes much more complicated in view of the fact that the state problem is now represented

*Received by the editors January 27, 2008; accepted for publication (in revised form) December 8, 2008; published electronically April 29, 2009. This work was supported by grants IAA1075402 and IAA100750802 of the Czech Academy of Sciences (JH, MK, JVO), grant 201/07/0294 of the Grant Agency of the Czech Republic (PB), research projects MSM6198910027 (PB, RK) and MSM0021620839 (JH) of the Czech Ministry of Education, and the EU Commission in the Sixth Framework Program, project 30717 PLATO-N (MK).

http://www.siam.org/journals/siopt/20-1/71442.html

†Faculty of Electrical Engineering, VŠB-Technical University of Ostrava, 17. listopadu 15, 708 33 Ostrava-Poruba, Czech Republic (petr.beremlijski@vsb.cz).

‡Department of Numerical Mathematics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic (Jaroslav.Haslinger@mff.cuni.cz).

§School of Mathematics, University of Birmingham, Birmingham B15 2TT, UK and Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Pod vodárenskou věží 4, 182 08 Praha 8, Czech Republic (kocvara@maths.bham.ac.uk).

¶Department of Mathematics and Descriptive Geometry, VŠB-Technical University of Ostrava, 17. listopadu 15, 708 33 Ostrava-Poruba, Czech Republic (radek.kucera@vsb.cz).

‖Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Pod vodárenskou věží 4, 182 08 Praha 8, Czech Republic (outrata@utia.cas.cz).

by a nontrivial implicit variational inequality [6]. For the sake of simplicity we restrict ourselves to structures consisting of one deformable body unilaterally supported by a rigid foundation (the so-called Signorini problem) considering Coulomb friction on a common part. The paper is solely devoted to a finite-dimensional case resulting from an appropriate finite element approximation of our problem and, in particular, to sensitivity analysis. The continuous setting, however, is also presented in order to clarify the form of the resulting discrete model. Since our discrete design variables are given by control points of Bézier surfaces shaping contact zones, we deal with the classic boundary variation approach in shape optimization; i.e., topological changes of the structure are not admitted.

The two-dimensional (2D) case of this problem was studied by the authors in [2]; there we used the so-called implicit programming approach [16, 21] combined with the generalized differential calculus of Clarke [3]. One could definitely extend the approach from [2] to the 3D case by using a polyhedral approximation of the friction cone. It is, however, much more challenging to consider in the 3D model the right *nonpolyhedral* second-order (Lorentz) cone, but then the respective extension of the technique from [2] meets serious hurdles. This concerns both the numerical solution of the respective state problem as well as appropriate stability and sensitivity issues. To facilitate the computation of the subgradient information, needed in the used numerical method, we have thus invoked the results of [13] and exploited the substantially richer generalized differential calculus of Mordukhovich [17, 19]. This means that we compute now a matrix from the Clarke *generalized Jacobian* of the discretized state mapping via the *limiting (Mordukhovich) coderivative* of this mapping. This has enabled us to have an efficient treatment of coupled multifunctions arising on the right-hand side of the generalized equation defining the state mapping.

The outline of the paper is as follows. Section 2 is devoted to a brief description of the state problem, i.e., the contact problem with Coulomb friction in its original, infinite-dimensional formulation. In section 3 we describe its finite element discretization and introduce our shape optimization problem. Thereafter we present various properties of the discretized state mapping and end up with the proof of its *strong regularity*. Section 4 concerns the used implicit-programming method. In particular, it deals with the computation of Clarke's subgradients of the respective composite cost function which have to be supplied to the used algorithm of nonsmooth optimization. In this section we make use of several sophisticated rules of generalized differentiation. The first part of the last section 5 is devoted to the numerical solution of the state problem. In the second part we present the test examples.

Our notation is standard: $\bar{A}$ denotes the closure of a set A and, for a multifunction $\Phi : X \rightrightarrows Y, \operatorname{Gr}\Phi = \{(x, y) \in X \times Y \mid y \in \Phi(x)\}$ is the graph of Φ. $\mathbb{B}_R$ denotes a ball in $\mathbb{R}^n$ of radius R centered at the origin. For a vector $x \in \mathbb{R}^n$ and an index set $\mathcal{N} \subset \{1, 2, \ldots, n\}$, $x_{\mathcal{N}}$ denotes the subvector of x composed from the components x_i, $i \in \mathcal{N}$. For a Lipschitz single-valued mapping $F : \mathbb{R}^n \to \mathbb{R}^m$, $\bar{\partial}F(x)$ is the *generalized Jacobian* of Clarke, defined by

$$\bar{\partial}F(x) = \operatorname{conv}\left\{\lim_{i\to\infty} \nabla F(x_i) \mid x_i \overset{\Omega_F}{\to} x\right\},$$

where "conv" denotes the convex hull and Ω_F is the set of points at which F is differentiable. If $m = 1$, we speak of the *Clarke subdifferential*.

In section 4 we extensively use the following notions of the generalized differential calculus of Mordukhovich [18, 19].

Given a closed set $A \subset \mathbb{R}^n$ and a point $\bar{x} \in A$, we denote by $\widehat{N}_A(\bar{x})$ the *Fréchet (regular) normal cone* to A at $\bar{x}$, defined by

$$\widehat{N}_A(\overline{x}) = \left\{ x^* \in \mathbb{R}^n \mid \limsup_{x \xrightarrow{A} \overline{x}} \frac{\langle x^*, x - \overline{x} \rangle}{\|x - \overline{x}\|} \leq 0 \right\}.$$

The *limiting (Mordukhovich) normal cone* to A at $\overline{x}$, denoted by $N_A(\overline{x})$, is defined by

$$N_A(\overline{x}) := \operatorname*{Lim\,sup}_{x \xrightarrow{A} \overline{x}} \widehat{N}_A(x),$$

where "Lim sup" is the Kuratowski–Painlevé outer limit of sets (see [24]). If A is convex, then $N_A(\overline{x}) = \widehat{N}_A(\overline{x})$ amounts to the classic normal cone in the sense of convex analysis. We say that A is *normally regular* at $\bar{x}$, provided $N_A(\bar{x}) = \widehat{N}_A(\bar{x})$.

On the basis of the above notions, we can also describe the local behavior of multifunctions. Let $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a multifunction with closed graph and $(\overline{x}, \overline{y}) \in \operatorname{Gr} \Phi$. The multifunction $\widehat{D}^*\Phi(\overline{x}, \overline{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, defined by

$$\widehat{D}^*\Phi(\overline{x}, \overline{y})(y^*) := \{x^* \in \mathbb{R}^n \mid (x^*, -y^*) \in \widehat{N}_{\operatorname{Gr}\Phi}(\overline{x}, \overline{y})\},$$

is called the *regular coderivative* of Φ at $(\overline{x}, \overline{y})$. Analogously, the multifunction $D^*\Phi(\overline{x}, \overline{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, defined by

$$D^*\Phi(\overline{x}, \overline{y})(y^*) := \{x^* \in \mathbb{R}^n \mid (x^*, -y^*) \in N_{\operatorname{Gr}\Phi}(\overline{x}, \overline{y})\},$$

is called the *limiting (Mordukhovich) coderivative* of Φ at $(\overline{x}, \overline{y})$. These two coderivatives coincide whenever $\operatorname{Gr}\Phi$ is normally regular at $(\bar{x}, \bar{y})$. If Φ happens to be single-valued, we simply write $\widehat{D}^*\Phi(\overline{x})(D^*\Phi(\overline{x}))$. If Φ is continuously differentiable, then $\widehat{D}^*\Phi(\overline{x}) = D^*\Phi(\overline{x})$ amounts to the adjoint Jacobian.

In what follows the considered discretized mechanical equilibrium is modeled by a *generalized equation (GE)* of the form

$$0 \in G(x, y) + Q(y), \tag{1.1}$$

where $G[\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m]$ is continuously differentiable and $Q[\mathbb{R}^m \rightrightarrows \mathbb{R}^m]$ is a closed-graph multifunction. Consider the reference pair $(\bar{x}, \bar{y}) \in \operatorname{Gr} S$, where $S[\mathbb{R}^n \rightrightarrows \mathbb{R}^m]$ is defined by

$$S(x) = \{y \in \mathbb{R}^m | 0 \in G(x, y) + Q(y)\}.$$

In agreement with a slight generalization [4] of Robinson's concept of strong regularity from [23] we say that the GE (1.1) satisfies the *strong regularity condition* (SRC) at $(\bar{x}, \bar{y})$ (is *strongly regular* at $(\bar{x}, \bar{y})$), provided there exist neighborhoods $\mathcal{U}$ of 0 and $\mathcal{V}$ of $\bar{y}$ such that the mapping

$$v \mapsto \{y \in \mathcal{V} | v \in G(\bar{x}, \bar{y}) + \nabla_y G(\bar{x}, \bar{y})(y - \bar{y}) + Q(y)\}$$

is single-valued and Lipschitz on $\mathcal{U}$. It is well known ([4, Theorem 2.1]) that the strong regularity of (1.1) at $(\bar{x}, \bar{y})$ implies the existence of neighborhoods $\tilde{\mathcal{U}}$ of $\bar{x}$ and $\tilde{\mathcal{V}}$ of $\bar{y}$ such that the mapping

$$x \mapsto S(x) \cap \tilde{\mathcal{V}}$$

is single-valued and Lipschitz on $\tilde{\mathcal{U}}$.

2. Setting of the problem. We start with the definition of the state problem. Let $\hat{\Omega} = \mathcal{R} \times (0,c)$, $\mathcal{R} = (0,a) \times (0,b)$, be a block in $\mathbb{R}^3$, $a,b,c > 0$ given. By U_{ad} we denote a family of admissible functions, where

$$U_{ad} = \Big\{ \alpha \in C^{0,1}\left(\bar{\mathcal{R}}\right) \mid 0 \le \alpha \le C_0 \text{ in } \bar{\mathcal{R}},\ \|\alpha'\|_{\infty,\mathcal{R}} \le C_1, \quad C_2 \le \int_{\mathcal{R}} \alpha \, \mathrm{d}x_1 \mathrm{d}x_2 \le C_3 \Big\}; \tag{2.1}$$

i.e., the set U_{ad} contains all nonnegative, bounded, Lipschitz equicontinuous functions in $\bar{\mathcal{R}}$, satisfying an integral type constraint. Positive numbers C_0, C_1, C_2, and C_3 are chosen in such a way that $U_{ad} \neq \emptyset$. With any $\alpha \in U_{ad}$ we associate a subdomain $\Omega(\alpha) \subset \hat{\Omega}$:

$$\Omega(\alpha) = \{(x_1,x_2,x_3) \in \hat{\Omega} \mid x_3 \ge \alpha(x_1,x_2) \quad \forall (x_1,x_2) \in \mathcal{R}\}.$$

Functions $\alpha \in U_{ad}$ will play the role of *control variables* determining the shape of $\Omega(\alpha)$.

Let $\alpha \in U_{ad}$ be *fixed*, and consider an elastic body represented by $\Omega(\alpha)$. Its boundary $\partial\Omega(\alpha)$ is split into three nonempty, nonoverlapping parts $\Gamma_u(\alpha)$, $\Gamma_p(\alpha)$, and $\Gamma_c(\alpha)$: $\partial\Omega(\alpha) = \bar{\Gamma}_u(\alpha) \cup \bar{\Gamma}_p(\alpha) \cup \bar{\Gamma}_c(\alpha)$, where different boundary conditions will be prescribed. On $\Gamma_u(\alpha)$ the body is fixed, while surface tractions of density $P = (P_1,P_2,P_3)$ act on $\Gamma_p(\alpha)$. The body is *unilaterally* supported along $\Gamma_c(\alpha)$ by a rigid foundation represented by the half-space $\mathbb{R}^2 \times \mathbb{R}_-$, where

$$\Gamma_c(\alpha) = \{(x_1,x_2,x_3) \mid x_3 = \alpha(x_1,x_2) \quad \forall (x_1,x_2) \in \mathcal{R}\}$$

is the graph of α. Finally, $\Omega(\alpha)$ is subject to body forces of density $F = (F_1,F_2,F_3)$. Our aim is to find an equilibrium state taking into account friction on $\Gamma_c(\alpha)$. This state is characterized by a displacement field $u = (u_1,u_2,u_3)$ satisfying the following system of differential equations and boundary conditions:

- *equilibrium equations*:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + F_i = 0 \text{ in } \Omega(\alpha),\ i = 1,2,3;^{1} \tag{2.2}$$

- *Hooke's law*:

$$\sigma_{ij} := \sigma_{ij}(u) = c_{ijkl}\epsilon_{kl}(u) \text{ in } \Omega(\alpha), \text{ where } \epsilon_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right), \quad i,j,k,l = 1,2,3; \tag{2.3}$$

- *kinematical boundary conditions*:

$$u_i = 0 \text{ on } \Gamma_u(\alpha),\ i = 1,2,3; \tag{2.4}$$

- *prescribed tractions*:

$$T_i := \sigma_{ij}\nu_j = P_i \text{ on } \Gamma_p(\alpha),\ i = 1,2,3; \tag{2.5}$$

[1] Here and in what follows the summation convention is adopted.

- *unilateral conditions*:

$$\left.\begin{aligned} &u_3(x', \alpha(x')) \geq -\alpha(x') \quad \forall x' = (x_1, x_2) \in \mathcal{R}; \\ &T_3(x) := \sigma_{33}(x) \geq 0,\ T_3(x)(u_3(x) + \alpha(x')) = 0 \quad \forall x \in \Gamma_c(\alpha); \end{aligned}\right\} \tag{2.6}$$

- *Coulomb law of friction*:

$$\left.\begin{aligned} &\text{if } u_t(x) := (u_1(x), u_2(x), 0) = 0 \Rightarrow \|T_t(x)\| \leq \mathcal{F}T_3(x), \\ &\qquad \text{where } T_t(x) := (T_1(x), T_2(x), 0),\ x \in \Gamma_c(\alpha); \\ &\text{if } u_t(x) \neq 0 \Rightarrow T_t(x) = -\mathcal{F}T_3(x)\frac{u_t(x)}{\|u_t(x)\|},\ x \in \Gamma_c(\alpha). \end{aligned}\right\} \tag{2.7}$$

Remark 2.1. The equations and boundary conditions (2.2)–(2.7) represent the *classical* formulation of a contact problem with Coulomb friction. The meaning of symbols is the following: $\sigma = (\sigma_{ij})_{i,j=1}^3$ stands for a symmetric stress tensor which is related to a linearized strain tensor $\epsilon = (\epsilon_{ij})_{i,j=1}^3$ by means of a linear Hooke's law (2.3), ν is the unit outward normal vector to $\partial\Omega(\alpha)$, and $T = (T_1, T_2, T_3)$ denotes the stress vector on $\partial\Omega(\alpha)$. Finally, $\|\ \|$ is the Euclidean norm of a vector and $\mathcal{F}$ is a coefficient of Coulomb friction. In what follows we shall suppose that $\mathcal{F}$ is a *positive constant*.

To give a weak form of the state problem we introduce the following spaces and sets:

$$\begin{aligned} V(\alpha) &= \{v = (v_1, v_2, v_3) \in (H^1(\Omega(\alpha)))^3 \mid v = 0 \text{ on } \Gamma_u(\alpha)\}, \\ K(\alpha) &= \{v \in V(\alpha) \mid v_3(x', \alpha(x')) \geq -\alpha(x') \text{ a.e. in } \mathcal{R}\}, \\ X &= \{\varphi \in L^2(\mathcal{R}) \mid \exists v \in V(\alpha) :\ \varphi(x') = v_3(x', \alpha(x')),\ x' \in \mathcal{R}\}, \\ &X' \text{ is the dual of } X,\ X'_+ \text{ is the cone of positive elements of } X'. \end{aligned}$$

The duality pairing between X' and X will be denoted by $\langle \cdot, \cdot \rangle$.

We start with a simpler model of friction in which the unknown component T_3 in (2.7) is replaced by a given slip bound $g \in X'_+$ (model with given friction; see [9]). Its mathematical formulation leads to a *variational inequality* of the 2nd kind:

$$\left.\begin{aligned} &\text{Find } u := u(g) \in K(\alpha) \text{ such that} \\ &\quad a_\alpha(u, v - u) + \langle \mathcal{F}g, \|\hat{v}_t\| - \|\hat{u}_t\| \rangle \geq L_\alpha(v - u) \quad \forall v \in K(\alpha), \end{aligned}\right\} \tag{$\mathcal{P}(g)$}$$

where

$$a_\alpha(u, v) := \int_{\Omega(\alpha)} c_{ijkl}\epsilon_{ij}(u)\epsilon_{kl}(v)\ \mathrm{d}x, \tag{2.8}$$

$$L_\alpha(v) := \int_{\Omega(\alpha)} F_i v_i\ \mathrm{d}x + \int_{\Gamma_p(\alpha)} P_i v_i\ \mathrm{d}s \tag{2.9}$$

are the inner energy and the work of external forces, respectively. Further, $\|\hat{v}_t\|$ denotes the Euclidean norm of the vector $\hat{v}_t := v_t \circ \alpha$ given by $(v_1(x', \alpha(x')), v_2(x', \alpha(x')), 0)$, $x' \in \mathcal{R}$. Next we shall suppose that $F \in (L^2(\hat{\Omega}))^3$, $P \in (L^2(\partial\hat{\Omega}))^3$, and the linear elasticity coefficients c_{ijkl} in (2.3) satisfy the usual symmetry and ellipticity conditions [20]. It is well-known that under these assumptions problem $(\mathcal{P}(g))$

has a unique solution $u(g)$ for every $g \in X'_+$. To release the unilateral constraint $u(g) \in K(\alpha)$ we use an alternative formulation of $(\mathcal{P}(g))$ involving Lagrange multipliers:

$$(\mathcal{M}(g)) \quad \left.\begin{array}{l} \text{Find } (u,\lambda) := (u(g),\lambda(g)) \in V(\alpha) \times X'_+ \text{ such that} \\ \quad a_\alpha(u, v-u) + \langle \mathcal{F}g, \|\hat{v}_t\| - \|\hat{u}_t\| \rangle \geq L_\alpha(v-u) + \langle \lambda, \hat{v}_3 - \hat{u}_3 \rangle \; \forall v \in V(\alpha) \\ \quad \langle \mu - \lambda, \hat{u}_3 + \alpha \rangle \geq 0 \quad \forall \mu \in X'_+, \end{array}\right\}$$

where $\hat{v}_3(x') := v_3(x', \alpha(x'))$, $x' \in \mathcal{R}$. It is easy to show that problem $(\mathcal{M}(g))$ has a unique solution $(u(g), \lambda(g))$ for every $g \in X'_+$. This makes it possible to define a mapping $\Phi : \; X'_+ \longmapsto X'_+$ by

$$\Phi(g) = \lambda(g) \quad \forall g \in X'_+, \tag{2.10}$$

where $\lambda(g) \in X'_+$ is the second component of the solution to $(\mathcal{M}(g))$.

DEFINITION 2.2. *By a weak solution of a contact problem with Coulomb friction we call any pair* $(u, \lambda) \in V(\alpha) \times X'_+$ *satisfying*

$$(Q(\alpha)) \quad \left.\begin{array}{l} a_\alpha(u, v-u) + \langle \mathcal{F}\lambda, \|\hat{v}_t\| - \|\hat{u}_t\| \rangle \geq L_\alpha(v-u) + \langle \lambda, \hat{v}_3 - \hat{u}_3 \rangle \quad \forall v \in V(\alpha) \\ \langle \mu - \lambda, \hat{u}_3 + \alpha \rangle \geq 0 \quad \forall \mu \in X'_+. \end{array}\right\}$$

From this definition it follows that λ is a fixed point of Φ. The existence of such fixed points has been analyzed in [6].

So far the function $\alpha \in U_{ad}$ characterizing the shape of $\Omega(\alpha)$ has been fixed. Now we shall look at $\alpha \in U_{ad}$ as a control variable governing state problem $(Q(\alpha))$. Let $\mathcal{S}$ be the respective control-to-state mapping defined by

$$\mathcal{S}(\alpha) = (u(\alpha), \lambda(\alpha)) \quad \forall \alpha \in U_{ad}, \tag{2.11}$$

where $(u(\alpha), \lambda(\alpha))$ is a solution to $(Q(\alpha))$, and denote by $\mathcal{G}$ the graph of $\mathcal{S}$. Since $(Q(\alpha))$ may have more than one solution, the mapping $\mathcal{S}$ is set-valued in general.

Finally, let $J : \; \mathcal{G} \longmapsto \mathbb{R}^1$ be a cost functional. Suppose that there exists a minimizer $(\alpha^*, u(\alpha^*), \lambda(\alpha^*))$ of J on $\mathcal{G}$. Then $\Omega^* := \Omega(\alpha^*)$ will be called an *optimal domain*. Since we are interested in numerical realization of this problem, its discretization will be necessary.

To discretize the geometry we introduce a system U^h_{ad} containing functions which are uniquely determined by a finite number of degrees of freedom (e.g., Bézier surfaces). Admissible domains will be then given by $\Omega(\alpha_h)$, where $\alpha_h \in U^h_{ad}$. To discretize the state problem we use a finite element method. The classical Galerkin method together with appropriate discretizations $V_h(\alpha_h)$, $K_h(\alpha_h)$, and X_h and X'_h of $V(\alpha)$, $K(\alpha)$, and X and X', respectively, represent an efficient tool of solving frictional contact problems. Analogously to the continuous setting we introduce a discrete control-to-state mapping $\mathcal{S}_h$ defined on U^h_{ad} and its graph $\mathcal{G}_h$. The discrete shape optimization is then given by the minimization of J on $\mathcal{G}_h$. In the next section we present an algebraic form of this problem arising from a typical finite element approximation. From this the structure of the problem to be solved will be seen.

3. Algebraic form of contact problems with Coulomb friction. The aim of this section is to present the algebraic form of the discretized state problem and to analyze its basic properties. We will proceed analogously to the continuous setting.

Let the discretization parameter $h > 0$ be fixed. Next we shall use the following notation: $\|\cdot\|_s$, $\langle \cdot, \cdot \rangle_s$ stand for the Euclidean norm and the scalar product in $\mathbb{R}^s$,

respectively. In the frequent case $s = 2$ the subscript will be omitted. Every $\alpha_h \in U_{ad}^h$ will be uniquely characterized by a discrete design variable $\boldsymbol{\alpha} \in \mathbb{R}^d$, and U_{ad}^h will be identified with a set $\mathcal{U} \subset \mathbb{R}^d$. We shall suppose that $\mathcal{U}$ is a *convex*, *compact* subset of $\mathbb{R}^d$. Let $\{\varphi_i\}_{i=1}^n$ be a basis of a finite element space $V_h(\alpha_h)$, $\alpha_h \in U_{ad}^h$, and suppose that its dimension n does not depend on α_h. Then

$$a_{\alpha_h}(v_h, z_h) = \langle \boldsymbol{A}(\boldsymbol{\alpha})\boldsymbol{v}, \boldsymbol{z}\rangle_n,$$
$$L_{\alpha_h}(v_h) = \langle \boldsymbol{L}(\boldsymbol{\alpha}), \boldsymbol{v}\rangle_n \quad \forall v_h, z_h \in V_h(\alpha_h),$$

where $\boldsymbol{v} \in \mathbb{R}^n$ is the vector of coordinates of v_h with respect to $\{\varphi_i\}_{i=1}^n$ and $\boldsymbol{A}(\boldsymbol{\alpha}) \in \mathbb{R}^{n\times n}$, $\boldsymbol{L}(\boldsymbol{\alpha}) \in \mathbb{R}^n$ are a stiffness matrix and a load vector, respectively, both depending on $\boldsymbol{\alpha} \in \mathcal{U}$. The set of all $\boldsymbol{A}(\boldsymbol{\alpha}) \in \mathbb{R}^{n\times n}$, $\boldsymbol{\alpha} \in \mathcal{U}$, will be denoted by M. For the sake of consistency we will denote in what follows all finite-dimensional vectors and all matrices by bold letters. At mappings, however, this convention will not be applied strictly, and so most vector-valued and set-valued mappings are denoted by standard (nonbold) letters. We shall suppose that the following assumptions are satisfied:

(3.1) All matrices $\boldsymbol{A} \in M$ are symmetric and uniformly positive definite, i.e., there exists $\gamma > 0$ such that $\langle \boldsymbol{A}\boldsymbol{v}, \boldsymbol{v}\rangle_n \geq \gamma \|\boldsymbol{v}\|_n^2 \quad \forall \boldsymbol{v} \in \mathbb{R}^n \quad \forall \boldsymbol{A} \in M;$

(3.2) The mappings $\boldsymbol{A} : \mathcal{U} \longmapsto M, \boldsymbol{L} : \mathcal{U} \longmapsto \mathbb{R}^n$ are continuously differentiable on an open set containing $\mathcal{U}$.

We start with the algebraic form of the contact problem with given friction for *fixed* $\boldsymbol{\alpha} \in \mathcal{U}$. Let $\boldsymbol{g} \in \mathbb{R}_+^p$ $(p < n)$ be a given discrete *slip bound* (p is related to the number of all contact nodes, i.e., the nodes of a used partition of $\bar{\Omega}(\alpha_h)$ into finite elements which are placed on $\bar{\Gamma}_c(\alpha_h) \setminus \bar{\Gamma}_u(\alpha_h)$). The discretized total potential energy has the following form:

$$\mathcal{J}(\boldsymbol{v}) = \frac{1}{2}\langle \boldsymbol{v}, \boldsymbol{A}\boldsymbol{v}\rangle_n - \langle \boldsymbol{L}, \boldsymbol{v}\rangle_n + \mathcal{F}\langle \boldsymbol{g}, |T\boldsymbol{v}|\rangle_p, \tag{3.3}$$

with $\boldsymbol{A} := \boldsymbol{A}(\boldsymbol{\alpha})$, $\boldsymbol{L} := \boldsymbol{L}(\boldsymbol{\alpha})$ for some $\boldsymbol{\alpha} \in \mathcal{U}$ (since $\boldsymbol{\alpha} \in \mathcal{U}$ is now fixed, it will be omitted in the argument of $\boldsymbol{A}$ and $\boldsymbol{L}$). Further, $T : \mathbb{R}^n \longmapsto \mathbb{R}^{2p}$ is a linear mapping defined by $T\boldsymbol{v} := (T_1\boldsymbol{v}, T_2\boldsymbol{v}, \ldots, T_p\boldsymbol{v})$, $\boldsymbol{v} \in \mathbb{R}^n$, where $T_i\boldsymbol{v} \in \mathbb{R}^2$ is the tangential nodal displacement vector at the ith contact node. The symbol $|T\boldsymbol{v}| \in \mathbb{R}^p$ denotes a vector defined by

$$|T\boldsymbol{v}| := (\|T_1\boldsymbol{v}\|, \ldots, \|T_p\boldsymbol{v}\|).$$

Let $\mathbb{K}$ be a closed convex subset of $\mathbb{R}^n$:

$$\mathbb{K} := \{\boldsymbol{v} \in \mathbb{R}^n \mid \boldsymbol{N}\boldsymbol{v} \geq -\boldsymbol{\alpha}\}, \tag{3.4}$$

where $\boldsymbol{N} \in \mathbb{R}^{p\times n}$ is a matrix representation of the nonpenetration property. From a displacement vector $v \in \mathbb{R}^n$, $\boldsymbol{N}$ selects the normal components at the p contact points. Clearly, $\boldsymbol{N}$ has the following properties:

(a) It has full row rank.
(b) Each column contains at most one nonzero element.
(c) $\boldsymbol{N}(\mathbb{R}^n) = \boldsymbol{N}(\ker T)$.

From (a) it follows that

$$\exists \beta = \text{const.} > 0 \text{ such that } \sup_{\substack{\boldsymbol{v}\in\mathbb{R}^n \\ \boldsymbol{v}\neq 0}} \frac{\langle \boldsymbol{\mu}, \boldsymbol{N}\boldsymbol{v}\rangle_p}{\|\boldsymbol{v}\|_n} \geq \beta\|\boldsymbol{\mu}\|_p \quad \forall \boldsymbol{\mu} \in \mathbb{R}^p. \tag{3.5}$$

DEFINITION 3.1. *By a solution of a discrete contact problem with given friction we mean a vector* $\boldsymbol{u} \in \mathbb{K}$ *satisfying*

$$(\mathcal{P}(\boldsymbol{L},\boldsymbol{g})) \qquad \langle \boldsymbol{A}\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}\rangle_n + \mathcal{F}\langle \boldsymbol{g}, |T\boldsymbol{v}| - |T\boldsymbol{u}|\rangle_p \geq \langle \boldsymbol{L}, \boldsymbol{v}-\boldsymbol{u}\rangle_n \quad \forall \boldsymbol{v} \in \mathbb{K}.$$

Next we will analyze the dependence of $\boldsymbol{u}$ on $\boldsymbol{L}$ and $\boldsymbol{g}$ with $\boldsymbol{\alpha} \in \mathcal{U}$ being fixed. This is why we use notation $(\mathcal{P}(\boldsymbol{L},\boldsymbol{g}))$.

To conduct this analysis, we introduce an equivalent formulation of $(\mathcal{P}(\boldsymbol{L},\boldsymbol{g}))$ which involves Lagrange multipliers releasing the constraint $\boldsymbol{v} \in \mathbb{K}$. It reads as follows:

$$(\mathcal{M}(\boldsymbol{L},\boldsymbol{g})) \qquad \left.\begin{array}{l} \text{Find } (\boldsymbol{u},\boldsymbol{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^p_+ \text{ such that} \\ \quad \langle \boldsymbol{A}\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}\rangle_n + \mathcal{F}\langle \boldsymbol{g}, |T\boldsymbol{v}| - |T\boldsymbol{u}|\rangle_p \\ \qquad\qquad \geq \langle \boldsymbol{L}, \boldsymbol{v}-\boldsymbol{u}\rangle_n + \langle \boldsymbol{\lambda}, \boldsymbol{N}\boldsymbol{v} - \boldsymbol{N}\boldsymbol{u}\rangle_p \quad \forall \boldsymbol{v} \in \mathbb{R}^n \\ \quad \langle \boldsymbol{\mu}-\boldsymbol{\lambda}, \boldsymbol{N}\boldsymbol{u}+\boldsymbol{\alpha}\rangle_p \geq 0 \quad \forall \boldsymbol{\mu} \in \mathbb{R}^p_+. \end{array}\right\}$$

The first component $\boldsymbol{u}$ of the solution to $(\mathcal{M}(\boldsymbol{L},\boldsymbol{g}))$ solves $(\mathcal{P}(\boldsymbol{L},\boldsymbol{g}))$, and the second component $\boldsymbol{\lambda}$ represents the discrete normal contact stress. The following result is easy to prove.

PROPOSITION 3.2. *Problems* $(\mathcal{P}(\boldsymbol{L},\boldsymbol{g}))$, $(\mathcal{M}(\boldsymbol{L},\boldsymbol{g}))$ *have unique solutions* $\boldsymbol{u}$, $(\boldsymbol{u},\boldsymbol{\lambda})$, *respectively, for every* $\boldsymbol{L} \in \mathbb{R}^n$ *and* $\boldsymbol{g} \in \mathbb{R}^p_+$.

Proof. Both existence and uniqueness follow directly from (3.1) and (3.5). ∎

Next we will establish several useful properties of the solution to $(\mathcal{M}(\boldsymbol{L},\boldsymbol{g}))$.

PROPOSITION 3.3. *Let* $(\boldsymbol{u},\boldsymbol{\lambda})$ *be the solution to* $(\mathcal{M}(\boldsymbol{L},\boldsymbol{g}))$. *Then*

$$\|\boldsymbol{u}\|_n \leq \frac{1}{\gamma}\|\boldsymbol{L}\|_n, \tag{3.6}$$

$$\|\boldsymbol{\lambda}\|_m \leq \frac{1}{\beta}\left(\frac{\|\boldsymbol{A}\|}{\gamma}+1\right)\|\boldsymbol{L}\|_n, \tag{3.7}$$

where γ, $\beta > 0$ *are from* (3.1) *and* (3.5).

Proof. Inserting $\boldsymbol{v} = \boldsymbol{0} \in \mathbb{K}$ into $(\mathcal{P}(\boldsymbol{L},\boldsymbol{g}))$ we obtain

$$\gamma\|\boldsymbol{u}\|_n^2 \leq \langle \boldsymbol{A}\boldsymbol{u}, \boldsymbol{u}\rangle_n + \mathcal{F}\langle \boldsymbol{g}, |T\boldsymbol{u}|\rangle_p \leq \langle \boldsymbol{L}, \boldsymbol{u}\rangle_n$$

from which (3.6) follows. Further, substitutions $\boldsymbol{v} = \boldsymbol{0}$ and $\boldsymbol{v} = 2\boldsymbol{u}$ into the first inequality in $(\mathcal{M}(\boldsymbol{L},\boldsymbol{g}))$ yield

$$\langle \boldsymbol{A}\boldsymbol{u}, \boldsymbol{u}\rangle_n + \mathcal{F}\langle \boldsymbol{g}, |T\boldsymbol{u}|\rangle_p = \langle \boldsymbol{L}, \boldsymbol{u}\rangle_n + \langle \boldsymbol{\lambda}, \boldsymbol{N}\boldsymbol{u}\rangle_p.$$

Therefore,

$$\langle \boldsymbol{A}\boldsymbol{u}, \boldsymbol{v}\rangle_n + \mathcal{F}\langle \boldsymbol{g}, |T\boldsymbol{v}|\rangle_p \geq \langle \boldsymbol{L}, \boldsymbol{v}\rangle_n + \langle \boldsymbol{\lambda}, \boldsymbol{N}\boldsymbol{v}\rangle_p \quad \forall \boldsymbol{v} \in \mathbb{R}^n. \tag{3.8}$$

From (3.8) it follows that

$$\langle \boldsymbol{A}\boldsymbol{u}, \boldsymbol{v}\rangle_n = \langle \boldsymbol{L}, \boldsymbol{v}\rangle_n + \langle \boldsymbol{\lambda}, \boldsymbol{N}\boldsymbol{v}\rangle_p \quad \forall \boldsymbol{v} \in \ker T. \tag{3.9}$$

From this and (3.5) we obtain (3.7) using that $\boldsymbol{N}(\mathbb{R}^n) = \boldsymbol{N}(\ker T)$. ∎

Remark 3.4. It is worth noticing that the bounds (3.6) and (3.7) do not depend on $\mathcal{F} > 0$ and $\boldsymbol{g} \in \mathbb{R}^p_+$.

Assume now that $\boldsymbol{g}$ is fixed, and let $\Psi : \mathbb{R}^n \longmapsto \mathbb{K} \times \mathbb{R}^p_+$ be a mapping defined by

$$\Psi(\boldsymbol{L}) = (\boldsymbol{u},\boldsymbol{\lambda}),$$

where $(\boldsymbol{u}, \boldsymbol{\lambda})$ is a solution of $(\mathcal{M}(\boldsymbol{L}, \boldsymbol{g}))$. It is very easy to show that Ψ is Lipschitz on $\mathbb{R}^n$ as follows from the next proposition.

PROPOSITION 3.5. *Let $(\boldsymbol{u}_i, \boldsymbol{\lambda}_i)$ be the solution of $(\mathcal{M}(\boldsymbol{L}_i, \boldsymbol{g}))$, $i = 1, 2$. Then*

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_n \leq \frac{1}{\gamma}\|\boldsymbol{L}_1 - \boldsymbol{L}_2\|_n, \tag{3.10}$$

$$\|\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2\|_p \leq \frac{1}{\beta}\left(\frac{\|\boldsymbol{A}\|}{\gamma} + 1\right)\|\boldsymbol{L}_1 - \boldsymbol{L}_2\|_n. \tag{3.11}$$

Proof. The proof can be done in the same way as the proof of Proposition 3.3. ∎

Next, to define a solution of a discrete contact problem with Coulomb friction, we fix $\boldsymbol{L}$ and introduce the mapping $\Gamma: \mathbb{R}^p_+ \longmapsto \mathbb{R}^p_+$:

$$\Gamma(\boldsymbol{g}) = \boldsymbol{\lambda}, \tag{3.12}$$

where $\boldsymbol{\lambda}$ is the second component of a solution to $(\mathcal{M}(\boldsymbol{L}, \boldsymbol{g}))$.

DEFINITION 3.6. *As a solution of a discrete contact problem with Coulomb friction we declare any couple $(\boldsymbol{u}, \boldsymbol{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^p_+$ satisfying*

$$(\mathcal{P}) \qquad \left.\begin{array}{l} \langle \boldsymbol{A}\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}\rangle_n + \mathcal{F}\langle \boldsymbol{\lambda}, |T\boldsymbol{v}| - |T\boldsymbol{u}|\rangle_p \\ \qquad\qquad \geq \langle \boldsymbol{L}, \boldsymbol{v} - \boldsymbol{u}\rangle_n + \langle \boldsymbol{\lambda}, \boldsymbol{N}\boldsymbol{v} - \boldsymbol{N}\boldsymbol{u}\rangle_p \quad \forall \boldsymbol{v} \in \mathbb{R}^n \\ \langle \boldsymbol{\mu} - \boldsymbol{\lambda}, \boldsymbol{N}\boldsymbol{u} + \boldsymbol{\alpha}\rangle_p \geq 0 \quad \forall \boldsymbol{\mu} \in \mathbb{R}^p_+; \end{array}\right\}$$

i.e., $\boldsymbol{\lambda}$ is a fixed point of Γ and $(\boldsymbol{u}, \boldsymbol{\lambda})$ solves $(\mathcal{M}(\boldsymbol{L}, \boldsymbol{\lambda}))$.

PROPOSITION 3.7. *For any $\boldsymbol{L} \in \mathbb{R}^n$ and any $\mathcal{F} > 0$ there exists at least one fixed point $\boldsymbol{\lambda}$ of Γ. All fixed points $\boldsymbol{\lambda}$ belong to $\mathbb{R}^p_+ \cap \mathbb{B}_R$, where $R = \frac{1}{\beta}(\frac{\|\boldsymbol{A}\|}{\gamma} + 1)\|\boldsymbol{L}\|_n$.*

Proof. It is easy to show that Γ is continuous in $\mathbb{R}^p_+$ and maps $\mathbb{R}^p_+ \cap \mathbb{B}_R$ into itself as follows from (3.7). The existence of a fixed point follows from Brower's fixed point theorem. ∎

Now we show that Γ is even contractive in $\mathbb{R}^p_+ \cap \mathbb{B}_R$, provided that $\mathcal{F}$ is small enough. Indeed, denote by $\boldsymbol{u}_i \in \mathbb{K}$ solutions to $(\mathcal{P}(\boldsymbol{L}, \boldsymbol{g}_i))$, $i = 1, 2$. Then one has

$$\langle \boldsymbol{A}\boldsymbol{u}_i, \boldsymbol{v} - \boldsymbol{u}_i\rangle_n + \mathcal{F}\langle \boldsymbol{g}_i, |T\boldsymbol{v}| - |T\boldsymbol{u}_i|\rangle_p \geq \langle \boldsymbol{L}, \boldsymbol{v} - \boldsymbol{u}_i\rangle_n \quad \forall \boldsymbol{v} \in \mathbb{K}. \tag{3.13}$$

Inserting $\boldsymbol{v} := \boldsymbol{u}_2$ into $(3.13)_1$ and $\boldsymbol{v} := \boldsymbol{u}_1$ into $(3.13)_2$ and summing up these inequalities we obtain

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_n \leq \frac{\mathcal{F}}{\gamma}\|T\|\|\boldsymbol{g}_1 - \boldsymbol{g}_2\|_p. \tag{3.14}$$

From (3.9) we know that

$$\langle \boldsymbol{A}\boldsymbol{u}_i, \boldsymbol{v}\rangle_n = \langle \boldsymbol{L}, \boldsymbol{v}\rangle_n + \langle \boldsymbol{\lambda}_i, \boldsymbol{N}\boldsymbol{v}\rangle_p \quad \forall \boldsymbol{v} \in \ker T.$$

Hence,

$$\|\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2\|_p \leq \frac{1}{\beta}\frac{\mathcal{F}}{\gamma}\|\boldsymbol{A}\|\|T\|\|\boldsymbol{g}_1 - \boldsymbol{g}_2\|_p. \tag{3.15}$$

In this way we obtain the following theorem.

THEOREM 3.8. *Let*

$$0 < \mathcal{F} < \frac{\beta\gamma}{\|\boldsymbol{A}\|\|T\|}. \tag{3.16}$$

Then Γ *defined by* (3.12) *is contractive in* $\mathbb{R}^p_+ \cap \mathbb{B}_R$ *so that the discrete contact problem with Coulomb friction has a unique solution.*

COROLLARY 3.9. *Let* (3.16) *be satisfied. Then the method of successive approximations*

$\boldsymbol{\lambda}^{(0)} \in \mathbb{R}^p_+$ *given*;
for $k = 0, 1, \ldots,$ *set* $\boldsymbol{\lambda}^{(k+1)} = \Gamma(\boldsymbol{\lambda}^{(k)})$ *until a stopping criterion is fulfilled*

is convergent to the unique fixed point of Γ *for any choice of* $\boldsymbol{\lambda}^{(0)} \in \mathbb{R}^p_+$.

Indeed, it follows from Proposition 3.3 that $\boldsymbol{\lambda}^{(1)} \in \mathbb{B}_R$ independent of the starting value $\boldsymbol{\lambda}^{(0)}$.

Remark 3.10. Let us notice that the bound (3.16) does not depend on $\boldsymbol{L} \in \mathbb{R}^n$ and it can be also chosen to be independent of $\boldsymbol{\alpha} \in \mathcal{U}$ as follows from (3.1); i.e., there exists an $\mathcal{F}_0 > 0$ such that for any $(\mathcal{F}, \boldsymbol{L}, \boldsymbol{\alpha}) \in (0, \mathcal{F}_0) \times \mathbb{R}^n \times \mathcal{U}$ there is a unique fixed point $\boldsymbol{\lambda}$ of Γ.

Let $\boldsymbol{\alpha} \in \mathcal{U}$ be fixed, $\mathcal{F} \in (0, \mathcal{F}_0)$, and $(\boldsymbol{u}, \boldsymbol{\lambda})$ be the unique solution of the contact problem with Coulomb friction in the sense of Definition 3.6. Now we shall consider the couple $(\boldsymbol{u}, \boldsymbol{\lambda}) := (\boldsymbol{u}(\boldsymbol{L}), \boldsymbol{\lambda}(\boldsymbol{L}))$ to be a function of the load vector $\boldsymbol{L} \in \mathbb{R}^n$. We define the mapping $S_L : \mathbb{R}^n \longmapsto \mathbb{K} \times \mathbb{R}^p_+$ by

$$S_L(\boldsymbol{L}) = (\boldsymbol{u}(\boldsymbol{L}), \boldsymbol{\lambda}(\boldsymbol{L})) \quad \forall \boldsymbol{L} \in \mathbb{R}^n.$$

An easy consequence of (3.11) and (3.14) is the fact that S_L is Lipschitz on $\mathbb{R}^n$ uniformly with respect to $\boldsymbol{\alpha} \in \mathcal{U}$.

PROPOSITION 3.11. *Let* $\mathcal{F} \in (0, \mathcal{F}_0)$. *Then there exists* $\delta > 0$ *such that the inequality*

$$\|S_L(\boldsymbol{L}_1) - S_L(\boldsymbol{L}_2)\|_{n+p} \leq \delta \|\boldsymbol{L}_1 - \boldsymbol{L}_2\|_n$$

holds for all $\boldsymbol{L}_1, \boldsymbol{L}_2 \in \mathbb{R}^n$ *and all* $\boldsymbol{\alpha} \in \mathcal{U}$.

So far, $\boldsymbol{\alpha} \in \mathcal{U}$ has been fixed. Next we will look at $\boldsymbol{\alpha}$ as a control variable parameterizing our state problem. We denote by S the *control-to-state mapping* which assigns $\boldsymbol{\alpha} \in \mathbb{R}^d$ the solutions $(\boldsymbol{u}, \boldsymbol{\lambda})$ of the contact problem with Coulomb friction in the sense of Definition 3.6. We know that $S(\boldsymbol{\alpha})$ is nonempty for all $\boldsymbol{\alpha} \in \mathcal{U}$ and a singleton if $\mathcal{F} \in (0, \mathcal{F}_0)$. Let $J : \operatorname{Gr} S \longmapsto \mathbb{R}$ be an objective function.

The *discrete optimal shape design problem* reads as follows:

$$(\mathbb{P}) \qquad \left.\begin{array}{c} \text{Find } \boldsymbol{z}^* := (\boldsymbol{\alpha}^*, \boldsymbol{u}^*, \boldsymbol{\lambda}^*) \in \operatorname{Gr} S \text{ such that} \\ J(\boldsymbol{z}^*) \leq J(\boldsymbol{z}) \quad \forall \boldsymbol{z} \in \operatorname{Gr} S. \end{array}\right\}$$

If $\mathcal{F} \in (0, \mathcal{F}_0)$, then S is single-valued and $(\mathbb{P})$ takes the form

$$(\tilde{\mathbb{P}}) \qquad \left.\begin{array}{c} \text{find } \boldsymbol{\alpha}^* \in \mathcal{U} \text{ such that} \\ \Theta(\boldsymbol{\alpha}^*) \leq \Theta(\boldsymbol{\alpha}) \quad \forall \boldsymbol{\alpha} \in \mathcal{U}, \end{array}\right\}$$

where $\Theta(\boldsymbol{\alpha}) := J(\boldsymbol{\alpha}, S(\boldsymbol{\alpha}))$. The following existence result is easy to prove.

THEOREM 3.12. *Let J be lower semicontinuous on* $\operatorname{Gr} S$. *Then* $(\mathbb{P})$ *has a solution.*

Proof. It is readily seen that $\operatorname{Gr} S$ is a compact subset of $\mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^m$. ∎

From now on we will suppose that $\mathcal{F} \in (0, \mathcal{F}_0)$. Our main aim is to show that the single-valued function S is Lipschitz on $\mathcal{U}$. We start with a reduction of the problem by eliminating all components of $\boldsymbol{u} \in \mathbb{R}^n$ associated with the noncontact nodes of the finite element partition of $\bar{\Omega}(\alpha_h)$. The resulting problem contains only the tangential components $\boldsymbol{u}_t \in \mathbb{R}^{2p}$ and the normal components $\boldsymbol{u}_\nu \in \mathbb{R}^p$ of $\boldsymbol{u}$ at p contact nodes. In the next step we transform this reduced problem into the following GE for the unknowns $\boldsymbol{u}_t$, $\boldsymbol{u}_\nu$, and $\boldsymbol{\lambda}$ (for details see section 3 in [2]):

$$\left.\begin{aligned} \mathbf{0} &\in \boldsymbol{A}_{tt}(\boldsymbol{\alpha})\boldsymbol{u}_t + \boldsymbol{A}_{t\nu}(\boldsymbol{\alpha})\boldsymbol{u}_\nu - \boldsymbol{L}_t(\boldsymbol{\alpha}) + Q_t(\boldsymbol{u}_t, \boldsymbol{\lambda}) \\ \mathbf{0} &= \boldsymbol{A}_{\nu t}(\boldsymbol{\alpha})\boldsymbol{u}_t + \boldsymbol{A}_{\nu\nu}(\boldsymbol{\alpha})\boldsymbol{u}_\nu - \boldsymbol{L}_\nu(\boldsymbol{\alpha}) - \boldsymbol{\lambda} \\ \mathbf{0} &\in \boldsymbol{u}_\nu + \boldsymbol{\alpha} + N_{\mathbb{R}^p_+}(\boldsymbol{\lambda}), \end{aligned}\right\} \tag{3.17}$$

where $\boldsymbol{A}_{tt}(\boldsymbol{\alpha}) \in \mathbb{R}^{2p\times 2p}$, $\boldsymbol{A}_{\nu\nu}(\boldsymbol{\alpha}) \in \mathbb{R}^{p\times p}$, $\boldsymbol{A}_{\nu t}(\boldsymbol{\alpha}) \in \mathbb{R}^{p\times 2p}$, $\boldsymbol{A}_{t\nu}(\boldsymbol{\alpha}) = \boldsymbol{A}^T_{\nu t}(\boldsymbol{\alpha})$, $\boldsymbol{L}_t(\boldsymbol{\alpha}) \in \mathbb{R}^{2p}$, and $\boldsymbol{L}_\nu(\boldsymbol{\alpha}) \in \mathbb{R}^p$.

In (3.17) $N_{\mathbb{R}^p_+}(\cdot)$ is the normal cone mapping in the sense of convex analysis and $Q_t : \mathbb{R}^{2p} \times \mathbb{R}^p \longmapsto \mathbb{R}^{2p}$ is the set-valued mapping defined by

$$Q_t(\boldsymbol{u}_t, \boldsymbol{\lambda}) = \partial_{u_t} j(\boldsymbol{u}_t, \boldsymbol{\lambda}), \;\; j(\boldsymbol{u}_t, \boldsymbol{\lambda}) = \mathcal{F} \sum_{i=1}^{p} \lambda_i \left\| \boldsymbol{u}_t^i \right\|,$$

with $\boldsymbol{u}_t^i \in \mathbb{R}^2$ being the tangential displacement at the ith contact node, $\boldsymbol{u}_t = \left(\boldsymbol{u}_t^1, \ldots, \boldsymbol{u}_t^p\right)$. Problem (3.17) can be written in a more compact form:

$$\mathbf{0} \in F(\boldsymbol{\alpha})\boldsymbol{y} - l(\boldsymbol{\alpha}) + Q(\boldsymbol{y}), \tag{3.18}$$

where $\boldsymbol{y} = (\boldsymbol{u}_t, \boldsymbol{u}_\nu, \boldsymbol{\lambda})$ and

$$F(\boldsymbol{\alpha}) = \begin{bmatrix} \boldsymbol{A}_{tt}(\boldsymbol{\alpha}) & \boldsymbol{A}_{t\nu}(\boldsymbol{\alpha}) & \mathbf{0} \\ \boldsymbol{A}_{\nu t}(\boldsymbol{\alpha}) & \boldsymbol{A}_{\nu\nu}(\boldsymbol{\alpha}) & -\mathbf{E} \\ \mathbf{0} & \mathbf{E} & \mathbf{0} \end{bmatrix},$$

$$l(\boldsymbol{\alpha}) = \left(\boldsymbol{L}_t(\boldsymbol{\alpha}), \boldsymbol{L}_\nu(\boldsymbol{\alpha}), -\boldsymbol{\alpha}\right)^T,$$

$$Q(\boldsymbol{y}) = \left(Q_t(\boldsymbol{u}_t, \boldsymbol{\lambda}), \mathbf{0}, N_{\mathbb{R}^p_+}(\boldsymbol{\lambda})\right)^T,$$

with $\mathbf{E}$ being the identity matrix.

In the rest of this section we prove that the GE (3.18) satisfies the SRC at $(\boldsymbol{\alpha}, \boldsymbol{y}(\boldsymbol{\alpha}))$ for any $\boldsymbol{\alpha} \in \mathcal{U}$, where $\boldsymbol{y}(\boldsymbol{\alpha})$ is the unique solution of (3.18). Let $(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{y}})$ be a reference point. We associate with (3.18) the following perturbed model at $(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{y}})$:

$$\boldsymbol{\xi} \in F(\bar{\boldsymbol{\alpha}})\boldsymbol{y} - l(\bar{\boldsymbol{\alpha}}) + Q(\boldsymbol{y}), \tag{3.19}$$

where $\boldsymbol{\xi} = (\boldsymbol{\xi}_t, \boldsymbol{\xi}_\nu, \boldsymbol{\xi}_\lambda) \in \mathbb{R}^{2p} \times \mathbb{R}^p \times \mathbb{R}^p$ is a canonical perturbation. This problem can be again interpreted as a contact problem with Coulomb friction with a *variable* load vector depending on $\boldsymbol{\xi}$ but formulated on a *given* shape specified by $\bar{\boldsymbol{\alpha}} \in \mathcal{U}$. Indeed, (3.19) is equivalent to

$$\left.\begin{aligned} \mathbf{0} &\in \boldsymbol{A}_{tt}(\bar{\boldsymbol{\alpha}})\boldsymbol{u}_t + \boldsymbol{A}_{t\nu}(\bar{\boldsymbol{\alpha}})\boldsymbol{u}_\nu - \boldsymbol{L}_t(\bar{\boldsymbol{\alpha}}) - \boldsymbol{\xi}_t + Q_t(\boldsymbol{u}_t, \boldsymbol{\lambda}) \\ \mathbf{0} &= \boldsymbol{A}_{\nu t}(\bar{\boldsymbol{\alpha}})\boldsymbol{u}_t + \boldsymbol{A}_{\nu\nu}(\bar{\boldsymbol{\alpha}})\boldsymbol{u}_\nu - \boldsymbol{L}_\nu(\bar{\boldsymbol{\alpha}}) - \boldsymbol{\xi}_\nu - \boldsymbol{\lambda} \\ \mathbf{0} &\in \boldsymbol{u}_\nu + \bar{\boldsymbol{\alpha}} + \boldsymbol{\xi}_\lambda + N_{\mathbb{R}^p_+}(\boldsymbol{\lambda}). \end{aligned}\right\} \tag{3.20}$$

The last inclusion in (3.20) corresponds to the nonpenetration condition $\boldsymbol{u}_\nu \geq -\bar{\boldsymbol{\alpha}} - \boldsymbol{\xi}_\lambda$. Denote by $\tilde{\boldsymbol{\xi}}_\lambda = (\boldsymbol{0}, \boldsymbol{\xi}_\lambda) \in \mathbb{R}^{2p}$ the extension of $\boldsymbol{\xi}_\lambda$ by the zero vector, and write $\tilde{\boldsymbol{u}} = (\boldsymbol{u}_t, \boldsymbol{u}_\nu)$ in the form

$$\tilde{\boldsymbol{u}} = \boldsymbol{w} - \tilde{\boldsymbol{\xi}}_\lambda, \tag{3.21}$$

i.e., $\boldsymbol{u}_t = \boldsymbol{w}_t$ and $\boldsymbol{u}_\nu = \boldsymbol{w}_\nu - \boldsymbol{\xi}_\lambda$, where $\boldsymbol{w}_\nu \geq -\bar{\boldsymbol{\alpha}}$. Inserting (3.21) into (3.20) we obtain a new problem for the vector $\boldsymbol{w} := \boldsymbol{w}(\boldsymbol{\xi})$:

$$\left.\begin{aligned} \boldsymbol{0} &\in \boldsymbol{A}_{tt}(\bar{\boldsymbol{\alpha}})\boldsymbol{w}_t + \boldsymbol{A}_{t\nu}(\bar{\boldsymbol{\alpha}})\boldsymbol{w}_\nu - \boldsymbol{F}_t(\boldsymbol{\xi}) + Q_t(\boldsymbol{w}_t, \boldsymbol{\lambda}) \\ \boldsymbol{0} &= \boldsymbol{A}_{\nu t}(\bar{\boldsymbol{\alpha}})\boldsymbol{w}_t + \boldsymbol{A}_{\nu\nu}(\bar{\boldsymbol{\alpha}})\boldsymbol{w}_\nu - \boldsymbol{F}_\nu(\boldsymbol{\xi}) - \boldsymbol{\lambda} \\ \boldsymbol{0} &\in \boldsymbol{w}_\nu + N_{\mathbb{R}^p_+}(\boldsymbol{\lambda}), \end{aligned}\right\} \tag{3.22}$$

where

$$\begin{aligned} \boldsymbol{F}_t(\boldsymbol{\xi}) &:= \boldsymbol{L}_t(\bar{\boldsymbol{\alpha}}) + \boldsymbol{\xi}_t + \boldsymbol{A}_{t\nu}(\bar{\boldsymbol{\alpha}})\boldsymbol{\xi}_\lambda, \\ \boldsymbol{F}_\nu(\boldsymbol{\xi}) &:= \boldsymbol{L}_\nu(\bar{\boldsymbol{\alpha}}) + \boldsymbol{\xi}_\nu + \boldsymbol{A}_{\nu\nu}(\bar{\boldsymbol{\alpha}})\boldsymbol{\xi}_\lambda \end{aligned}$$

represent a perturbation of the original load vector depending on $\boldsymbol{\xi}$. From Remark 3.4 we know that there exists a unique solution of (3.22) for any $\boldsymbol{\xi}$, and Proposition 3.11 says that this solution as a function of load vectors is Lipschitz on $\mathbb{R}^n$. Hence, we obtain the following result.

THEOREM 3.13. *Let $\mathcal{F} \in (0, \mathcal{F}_0)$ with $\mathcal{F}_0 > 0$ sufficiently small. Then the respective GE* (3.18) *is strongly regular at each triple $(\boldsymbol{\alpha}, \boldsymbol{u}, \boldsymbol{\lambda})$ with $\boldsymbol{\alpha} \in \mathcal{U}$ and $(\boldsymbol{u}, \boldsymbol{\lambda}) = S(\boldsymbol{\alpha})$. Consequently, the control-to-state mapping S is Lipschitz on $\mathcal{U}$.*

Proof. The proof follows from Theorem 2.1 in [4]. ∎

On the basis of the results of this section the Lipschitz continuity of S could be proved in a direct way, without using the mentioned Robinson's result. The strong regularity of (3.18) will play, however, an important role in the next section.

4. Implicit programming approach and sensitivity analysis. In this section the scalar product in $\mathbb{R}^n$ will be denoted by $\langle \cdot, \cdot \rangle$ without any subscript related to the dimension, $\{0\}_k$ means the zero vector in $\mathbb{R}^k$, and $\mathbb{B}$ stands for the unit ball. Consider the problem $(\tilde{\mathbb{P}})$, and assume that the objective J is continuously differentiable. By virtue of the assumptions and Theorem 3.13, Θ is locally Lipschitz on an open set containing $\mathcal{U}$. It follows that $(\tilde{\mathbb{P}})$ can be solved by a suitable bundle method of nondifferentiable optimization, provided the structure of $\mathcal{U}$ is not too complicated. To this end we have to be able to compute for each admissible design variable $\bar{\boldsymbol{\alpha}}$

(i) the solution $\bar{\boldsymbol{y}} = S(\bar{\boldsymbol{\alpha}})$ of the respective contact problem and

(ii) an arbitrary subgradient $\boldsymbol{\xi}$ from the Clarke subdifferential $\bar{\partial}\Theta(\bar{\boldsymbol{\alpha}})$.

This section is devoted to task (ii). We start with the observation that

$$\bar{\partial}\Theta(\bar{\boldsymbol{\alpha}}) = \nabla_{\boldsymbol{\alpha}} J(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{y}}) + \{\boldsymbol{C}^T \nabla_{\boldsymbol{y}} J(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{y}}) \mid \boldsymbol{C} \in \bar{\partial} S(\bar{\boldsymbol{\alpha}})\}$$

which follows directly from the chain rule in [3, Theorem 2.6.6]. Because of Lipschitz continuity of S one has from [17, Corollary 3.3.2] that $D^*S(\bar{\alpha})(y^*) \neq \emptyset$ for all y^* and from [18, formula (2.23)] that

$$\operatorname{conv}(D^*S(\bar{\alpha}))(y^*) = (\bar{\partial} S(\bar{\alpha}))^* y^*.$$

Thus, it suffices for our purposes to compute just one element from the set

$$D^*S(\bar{\boldsymbol{\alpha}})(\nabla_{\boldsymbol{y}} J(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{y}})),$$

and we are done. To be able to express the coderivative $D^*S(\bar{\boldsymbol{\alpha}})$ in terms of the data of the GE (3.18), we reorder this GE in such a way that the multivalued part

$$Q(\boldsymbol{y}) = \begin{bmatrix} \Phi(\boldsymbol{y}^1) \\ \Phi(\boldsymbol{y}^2) \\ \vdots \\ \Phi(\boldsymbol{y}^p) \end{bmatrix},$$

where the subvector $\boldsymbol{y}^i = (\boldsymbol{u}_t^i, u_\nu^i, \lambda^i) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}_+$ comprises all state variables associated with the ith contact node and

$$\Phi(\boldsymbol{y}^i) = \begin{bmatrix} \mathcal{F}\lambda^i \partial \|\boldsymbol{u}_t^i\| \\ 0 \\ N_{\mathbb{R}_+}(\lambda^i) \end{bmatrix},$$

$i = 1, 2, \ldots, p$. Let $\bar{\boldsymbol{d}} = (\bar{\boldsymbol{d}}^1, \bar{\boldsymbol{d}}^2, \ldots, \bar{\boldsymbol{d}}^p) \in (\mathbb{R}^4)^p$ belong to the image set $Q(\bar{\boldsymbol{y}})$ so that

$$\bar{\boldsymbol{d}}^i \in \Phi(\bar{\boldsymbol{y}}^i);\ i = 1, 2, \ldots, p.$$

Due to the above reordering, $\boldsymbol{y}^* = (\boldsymbol{y}^{*1}, \boldsymbol{y}^{*2}, \ldots, \boldsymbol{y}^{*p}) \in (\mathbb{R}^4)^p$ belongs to $D^*Q(\bar{\boldsymbol{y}}, \bar{\boldsymbol{d}})(\boldsymbol{d}^*)$ with $\boldsymbol{d}^* = (\boldsymbol{d}^{*1}, \boldsymbol{d}^{*2}, \ldots, \boldsymbol{d}^{*p}) \in (\mathbb{R}^4)^p$ whenever

$$\boldsymbol{y}^{*i} \in D^*\Phi(\bar{\boldsymbol{y}}^i, \bar{\boldsymbol{d}}^i)(\bar{\boldsymbol{d}}^{*i}),\ i = 1, 2, \ldots, p.$$

This facilitates the computation of the coderivative D^*Q conducted in the subsequent analysis. Put $m := 4p$.

THEOREM 4.1. *Consider the reference pair* $(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{y}})$ *with* $\bar{\boldsymbol{\alpha}} \in \mathcal{U}, \bar{\boldsymbol{y}} = S(\bar{\boldsymbol{\alpha}})$. *Then for each* $\boldsymbol{y}^* \in \mathbb{R}^m$

$$(\nabla_{\boldsymbol{\alpha}}(F(\bar{\boldsymbol{\alpha}})\bar{\boldsymbol{y}} - l(\bar{\boldsymbol{\alpha}})))^T \boldsymbol{z} \subset D^*S(\bar{\boldsymbol{\alpha}})(\boldsymbol{y}^*) \subset (\nabla_{\boldsymbol{\alpha}}(F(\bar{\boldsymbol{\alpha}})\bar{\boldsymbol{y}} - l(\bar{\boldsymbol{\alpha}})))^T \mathcal{V}, \tag{4.1}$$

provided $\boldsymbol{z}$ *is a solution of the* regular adjoint *GE (RAGE)*

$$0 \in \boldsymbol{y}^* + (F(\bar{\boldsymbol{\alpha}}))^T \boldsymbol{z} + \widehat{D}^*Q(\bar{\boldsymbol{y}}, -F(\bar{\boldsymbol{\alpha}})\bar{\boldsymbol{y}} + l(\bar{\boldsymbol{\alpha}}))(\boldsymbol{z}) \tag{4.2}$$

and $\mathcal{V}$ *is the set of solutions* $\boldsymbol{v}$ *to the* (limiting) adjoint *GE (AGE)*

$$0 \in \boldsymbol{y}^* + (F(\bar{\boldsymbol{\alpha}}))^T \boldsymbol{v} + D^*Q(\bar{\boldsymbol{y}}, -F(\bar{\boldsymbol{\alpha}})\bar{\boldsymbol{y}} + l(\bar{\boldsymbol{\alpha}}))(\boldsymbol{v}). \tag{4.3}$$

Proof. By Theorem 3.13 the GE (3.18) satisfies SRC at $(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{y}})$. It follows from [22, Proposition 3.2] that the qualification condition

$$\left.\begin{aligned} 0 &= (\nabla_{\boldsymbol{\alpha}}(F(\bar{\boldsymbol{\alpha}})\bar{\boldsymbol{y}} - l(\bar{\boldsymbol{\alpha}})))^T \boldsymbol{v} \\ 0 &\in (F(\bar{\boldsymbol{\alpha}}))^T \boldsymbol{v} + D^*Q(\bar{\boldsymbol{y}}, -F(\bar{\boldsymbol{\alpha}})\bar{\boldsymbol{y}} + l(\bar{\boldsymbol{\alpha}}))(\boldsymbol{v}) \end{aligned}\right\} \Rightarrow\ \boldsymbol{v} = 0,$$

is fulfilled as well. Thus, the result holds true by virtue of [13, Theorem 2]. ∎

COROLLARY 4.2. *If* $\operatorname{Gr} Q$ *is normally regular at* $(\bar{y}, -F(\bar{\boldsymbol{\alpha}})\bar{y} + l(\bar{\boldsymbol{\alpha}}))$, *then the second inclusion in* (4.1) *becomes equality.*

Let $\boldsymbol{y}^*$ be an arbitrary vector from $\mathbb{R}^m$. From the nonemptiness of $\operatorname{conv} D^*S(\bar{\boldsymbol{\alpha}})(\boldsymbol{y}^*)$ we conclude that AGE possesses a solution $\tilde{\boldsymbol{v}}$ for each $\boldsymbol{y}^* \in \mathbb{R}^m$. Further, again by the Lipschitz continuity of S, AGE has among its solutions at least one which belongs to the image of $\boldsymbol{y}^*$ by the linear mapping $(\lim_{i\to\infty}(\nabla S(\boldsymbol{\alpha}^{(i)}))^T(\cdot)$ for some sequence

TABLE 4.1

	no contact: $a_4 = 0, b_4 < 0$	weak contact: $a_4 = 0, b_4 = 0$	strong contact: $a_4 > 0, b_4 = 0$
sliding: $\boldsymbol{a}_{12} \neq 0$	L	M_2	M_1
weak sticking: $\boldsymbol{a}_{12} = 0$, $\|\boldsymbol{b}_{12}\| = \mathcal{F}a_4$		M_4	M_3^-
strong sticking: $\boldsymbol{a}_{12} = 0$, $\|\boldsymbol{b}_{12}\| < \mathcal{F}a_4$	- - - - - - -	- - - - - - -	M_3^+

$\boldsymbol{\alpha}^{(i)} \to \bar{\boldsymbol{\alpha}}$). Hence, at least one solution of AGE can be computed relatively easily. Under the normal regularity assumption of the corollary, RAGE and AGE coincide and so the above conclusions hold also for RAGE. In the nonregular case, however, RAGE may be difficult to solve or may even not be solvable at all. From these reasons we will use in the computation of the desired subgradient $\boldsymbol{\xi} \in \bar{\partial}\Theta(\bar{\boldsymbol{\alpha}})$ exclusively the way via the AGE (4.3) and accept that in the nonregular case the computed vector $\boldsymbol{\xi}$ may be outside of $\bar{\partial}\Theta(\bar{\boldsymbol{\alpha}})$. If this happens, the used bundle method need not inevitably collapse. If it does, however, $\boldsymbol{\xi}$ must be replaced by a *correct* subgradient.

In the next part of this section we will concentrate on the solution of AGE in the case of the GE (3.18) (after the performed reordering). Thereby the most important issue is the position of the point $(\bar{\boldsymbol{y}}, -F(\bar{\boldsymbol{\alpha}})\bar{\boldsymbol{y}} + l(\bar{\boldsymbol{\alpha}}))$ in $\operatorname{Gr} Q$ or, more precisely, the positions of $(\bar{\boldsymbol{y}}^i, -F^i(\bar{\boldsymbol{\alpha}})\bar{\boldsymbol{y}} + l^i(\bar{\boldsymbol{\alpha}}))$ in $\operatorname{Gr}\Phi$, $i = 1, 2, \ldots, p$, where $F^i(\bar{\boldsymbol{\alpha}})\bar{\boldsymbol{y}} - l^i(\bar{\boldsymbol{\alpha}})$ is the subvector of $F(\bar{\boldsymbol{\alpha}})\bar{\boldsymbol{y}} - l(\bar{\boldsymbol{\alpha}})$ associated with the ith contact node.

To facilitate the notation, let us consider instead of $(\boldsymbol{y}^i, -F^i(\boldsymbol{\alpha})\boldsymbol{y} + l^i(\bar{\boldsymbol{\alpha}}))$ a pair of vectors $(\boldsymbol{a}, \boldsymbol{b}) \in \operatorname{Gr}\Phi \subset \mathbb{R}^4 \times \mathbb{R}^4$ (so that, necessarily, $b_3 = 0$) and let us denote by $\boldsymbol{a}_{12}, \boldsymbol{b}_{12}$ the 2D vectors $(a_1, a_2)^T$, $(b_1, b_2)^T$, respectively.

Clearly, $\operatorname{Gr}\Phi$ admits the partition

$$\operatorname{Gr}\Phi = L \cup M_1 \cup M_2 \cup M_3^+ \cup M_3^- \cup M_4,$$

where the sets $L, M_1, M_2, M_3^+, M_3^-$, and M_4 are defined in Table 4.1.

The nature of these sets can be explained in mechanical terms. So, L refers to the absence of *contact* (and hence also the absence of friction). When $\boldsymbol{a}_{12} \neq 0$, we speak about *sliding*, in contrast to the case $\boldsymbol{a}_{12} = 0$ which is called *sticking*. The adjective "weak" is used as in complementarity [21] to indicate an unstable situation on the boundary between two stable states. All in all, M_1 amounts to sliding with contact, M_2 to sliding with weak contact, M_3^+ to sticking with contact, M_3^- to weak sticking with contact, and M_4 to weak sticking with weak contact. Impossible combinations are crossed out.

The sets L, M_1, and M_3^+ exhibit stable behavior in the sense that the implication

$$\left.\begin{array}{l}(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}}) \in L(\text{ or } M_1 \text{ or } M_3^+)\\ (\boldsymbol{a}, \boldsymbol{b}) \in \operatorname{Gr}\Phi\\ (\boldsymbol{a}, \boldsymbol{b}) \text{ is sufficiently close to } (\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}})\end{array}\right\} \Rightarrow\ (\boldsymbol{a}, \boldsymbol{b}) \in L(\text{ or } M_1 \text{ or } M_3^+)$$

is fulfilled. Indeed, e.g., in the case of M_3^+, the perturbed points $a_4, \boldsymbol{b}_{12}$ also satisfy the inequality $\|\boldsymbol{b}_{12}\| < \mathcal{F}a_4$, provided the perturbations are sufficiently small. It follows that $\boldsymbol{a}_{12}$ and b_4 cannot be perturbed at all; otherwise $(\boldsymbol{a}, \boldsymbol{b})$ would not be in $\operatorname{Gr}\Phi$.

So, let us start with the computation of the coderivative of Φ for the sets L, M_1, and M_3^+.

PROPOSITION 4.3. *Let* $(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}}) \in L$. *Then one has*

$$D^*\Phi(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}})(\boldsymbol{b}^*) = \begin{cases} \{0\}_3 \times \mathbb{R} & \textit{provided } b_4^* = 0, \\ \emptyset & \textit{otherwise.} \end{cases} \tag{4.4}$$

Proof. By the definitions of L and Φ, there is a neighborhood $\mathcal{O}$ of $(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}})$ such that

$$\operatorname{Gr}\Phi \cap \mathcal{O} = \{(\boldsymbol{a}, \boldsymbol{b}) | a_4 = b_1 = b_2 = b_3 = 0, b_4 < 0\} \cap \mathcal{O},$$

which means that $\operatorname{Gr}\Phi$ coincides locally around $(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}})$ with $(\mathbb{R}^3 \times \{0\}) \times (\{0\}_3 \times \mathbb{R})$, and so $N_{\operatorname{Gr}\Phi}(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}}) = (\{0\}_3 \times \mathbb{R}) \times (\mathbb{R}^3 \times \{0\})$. Now the result follows from the definition of the coderivative. ∎

PROPOSITION 4.4. *Let* $(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}}) \in M_1$. *Then one has*

$$D^*\Phi(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}})(\boldsymbol{b}^*) = \begin{bmatrix} & \boldsymbol{C}^T & & & 0 & 0 \\ & & & & 0 & 0 \\ 0 & & 0 & & 0 & 0 \\ \|\bar{\boldsymbol{a}}_{12}\|^{-1}\mathcal{F}\bar{a}_1 & & \|\bar{\boldsymbol{a}}_{12}\|^{-1}\mathcal{F}\bar{a}_2 & & 0 & 0 \end{bmatrix} \boldsymbol{b}^*, \tag{4.5}$$

where

$$\boldsymbol{C} = \frac{\mathcal{F}\bar{a}_4}{\|\bar{\boldsymbol{a}}_{12}\|^3} \begin{bmatrix} (\bar{a}_2)^2 & -\bar{a}_1\bar{a}_2 \\ -\bar{a}_1\bar{a}_2 & (\bar{a}_1)^2 \end{bmatrix}.$$

Proof. There is a neighborhood $\widetilde{\mathcal{O}}$ of $(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}})$ such that for all $(\boldsymbol{a}, \boldsymbol{b}) \in \operatorname{Gr}\Phi \cap \widetilde{\mathcal{O}}$ it holds that

$$\boldsymbol{b} = \mathcal{F} \begin{bmatrix} \|\boldsymbol{a}_{12}\|^{-1} a_4 a_1 \\ \|\boldsymbol{a}_{12}\|^{-1} a_4 a_2 \\ 0 \\ 0 \end{bmatrix},$$

and so it suffices to compute the adjoint Jacobian of this operator. ∎

PROPOSITION 4.5. *Let* $(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}}) \in M_3^+$. *Then one has*

$$D^*\Phi(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}})(\boldsymbol{b}^*) = \begin{cases} \mathbb{R}_2 \times \{0\}_2 & \textit{provided } b_1^* = b_2^* = 0, \\ \emptyset & \textit{otherwise.} \end{cases} \tag{4.6}$$

Proof. By the definitions of M_3^+ and Φ, there is a neighborhood $\widehat{\mathcal{O}}$ of $(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}})$ such that

$$\operatorname{Gr}\Phi \cap \widehat{\mathcal{O}} = \{(\boldsymbol{a}, \boldsymbol{b}) | a_1 = a_2 = 0, \|\boldsymbol{b}_{12}\| < \mathcal{F}a_4, b_3 = b_4 = 0\} \cap \widehat{\mathcal{O}}.$$

This means that $\operatorname{Gr}\Phi$ coincides locally around $(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}})$ with $(\{0\}_2 \times \mathbb{R}^2) \times (\mathbb{R}^2 \times \{0\}_2)$, and so $N_{\operatorname{Gr}\Phi}(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}}) = (\mathbb{R}^2 \times \{0\}_2) \times (\{0\}_2 \times \mathbb{R}^2)$. The rest follows again from the definition of the coderivative. ∎

Since $\operatorname{Gr}\Phi$ is normally regular at all points of $L \cup M_1 \cup M_3^+$, (4.4)–(4.6) lead to a correct subgradient $\boldsymbol{\xi}$ whenever all points $(\bar{\boldsymbol{y}}^i, -F^i(\bar{\boldsymbol{\alpha}})\bar{\boldsymbol{y}} + l^i(\bar{\boldsymbol{\alpha}})), i = 1, 2, \ldots, p$,

belong to this union. The situation is not so satisfactory, provided we have to do with the remaining sets M_2, M_3^-, and M_4. In the case of M_2 we can employ the following auxiliary statement.

LEMMA 4.6. *Consider the multifunction* $F : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^o \rightrightarrows \mathbb{R}^l \times \mathbb{R}^p$ *given by*

$$F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \begin{bmatrix} G(\boldsymbol{x}, \boldsymbol{y}) \\ H(\boldsymbol{y}, \boldsymbol{z}) \end{bmatrix},$$

where $G : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^l, H : \mathbb{R}^m \times \mathbb{R}^o \rightrightarrows \mathbb{R}^p$ *are closed-graph multifunctions. Assume that the point* $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, \bar{\boldsymbol{z}}, \bar{\boldsymbol{d}}_1, \bar{\boldsymbol{d}}_2)$ *belongs to* $\operatorname{Gr} F$ *and the qualification condition*

$$\left.\begin{array}{c} \begin{bmatrix} 0 \\ \boldsymbol{w}_2 \end{bmatrix} \in D^*G(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})(0) \\ \begin{bmatrix} -\boldsymbol{w}_2 \\ 0 \end{bmatrix} \in D^*H(\bar{\boldsymbol{y}}, \bar{\boldsymbol{z}})(0) \end{array}\right\} \Rightarrow \boldsymbol{w}_2 \Rightarrow 0 \tag{4.7}$$

holds true. Then one has

$$\begin{aligned} &D^*F(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, \bar{\boldsymbol{z}}, \bar{\boldsymbol{d}}_1, \bar{\boldsymbol{d}}_2)(\boldsymbol{d}_1^*, \boldsymbol{d}_2^*) \\ &\quad \subset \{(\boldsymbol{w}_1, \boldsymbol{w}_2 + \boldsymbol{w}_3, \boldsymbol{w}_4) \,|\, (\boldsymbol{w}_1, \boldsymbol{w}_2) \in D^*G(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})(\boldsymbol{d}_1^*), (\boldsymbol{w}_3, \boldsymbol{w}_4) \in D^*H(\bar{\boldsymbol{y}}, \bar{\boldsymbol{z}})(\boldsymbol{d}_2^*)\}. \end{aligned} \tag{4.8}$$

If G *happens to be continuously differentiable near* $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$*, then condition* (4.7) *is fulfilled and*

$$\begin{aligned} &D^*F(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, \bar{\boldsymbol{z}}, \bar{\boldsymbol{d}}_1, \bar{\boldsymbol{d}}_2)(\boldsymbol{d}_1^*, \boldsymbol{d}_2^*) \\ &\quad = \{((\nabla_{\boldsymbol{x}} G(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}))^T \boldsymbol{d}_1^*, (\nabla_{\boldsymbol{y}} G(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}))^T \boldsymbol{d}_1^* + \boldsymbol{w}_3, \boldsymbol{w}_4) \,|\, (\boldsymbol{w}_3, \boldsymbol{w}_4) \in D^*H(\bar{\boldsymbol{y}}, \bar{\boldsymbol{z}})(\boldsymbol{d}_2^*)\}. \end{aligned} \tag{4.9}$$

Proof. Clearly, F amounts to the composition $\begin{bmatrix} G \\ H \end{bmatrix} \circ A$, where $A : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^o \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^o$ is defined by

$$A = \begin{bmatrix} Id & 0 & 0 \\ 0 & Id & 0 \\ 0 & Id & 0 \\ 0 & 0 & Id \end{bmatrix}.$$

Inclusion (4.8) follows thus directly from [24, Theorem 10.40 (second statement)]. To prove the converse inclusion, observe first that under the differentiability of G the condition (4.7) is automatically fulfilled. Moreover, by the definition, each element $\boldsymbol{\xi}$ of the right-hand side of (4.8) can be expressed as

$$\boldsymbol{\xi} = \lim_{i \to \infty} A^T \begin{bmatrix} (\nabla G(\bar{\boldsymbol{x}}, \boldsymbol{y}^{(i)}))^T \boldsymbol{d}_1^* \\ \widehat{D}^* H(\boldsymbol{y}^{(i)}, \boldsymbol{z}^{(i)})(\boldsymbol{d}_2^{*(i)}) \end{bmatrix}$$

for some sequence $(\boldsymbol{y}^{(i)}, \boldsymbol{z}^{(i)}, \boldsymbol{d}_2^{*(i)}) \to (\bar{\boldsymbol{y}}, \bar{\boldsymbol{z}}, \boldsymbol{d}_2^*)$. By [24, Theorem 10.40 (first statement)], it holds for all i that

$$A^T \begin{bmatrix} (\nabla G(\bar{\boldsymbol{x}}, \boldsymbol{y}^{(i)}))^T (\boldsymbol{d}_1^*) \\ \widehat{D}^* H(\boldsymbol{y}^{(i)}, \boldsymbol{z}^{(i)})(\boldsymbol{d}_2^{*(i)}) \end{bmatrix} \subset \widehat{D}^* F(\bar{\boldsymbol{x}}, \boldsymbol{y}^{(i)}, \boldsymbol{z}^{(i)})(\boldsymbol{d}_1^*, \boldsymbol{d}_2^{*(i)}),$$

and so necessarily $\boldsymbol{\xi} \in D^*F(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, \bar{\boldsymbol{z}})(\boldsymbol{d}_1^*, \boldsymbol{d}_2^*)$. ∎

We observe now that in the case of $(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}})$ in M_2 the multifunction Φ possesses the structure of the map F in the above lemma with one of the multifunctions G, H differentiable. So, we can use the relation (4.9) in the following statement.

PROPOSITION 4.7. *Let* $(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}}) \in M_2$. *Then one has*

$$D^*\Phi(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}})(\boldsymbol{b}^*) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{\bar{a}_1}{\|\bar{\boldsymbol{a}}_{12}\|} b_1^* + \dfrac{\bar{a}_2}{\|\bar{\boldsymbol{a}}_{12}\|} b_2^* + w \end{bmatrix} \mid w \in \begin{cases} \mathbb{R} & \text{if } \mathrm{b}_4^* = 0 \\ \mathbb{R}_- & \text{if } \mathrm{b}_4^* < 0 \\ 0 & \text{otherwise} \end{cases} \right\}. \tag{4.10}$$

Proof. Since the mapping $\boldsymbol{a} \mapsto \mathcal{F} a_4 \frac{\boldsymbol{a}_{12}}{\|\bar{\boldsymbol{a}}_{12}\|}$ is continuously differentiable on a neighborhood of $\bar{\boldsymbol{a}}$, it suffices to compute the coderivative of the multifunction $N_{\mathbb{R}_+}(\cdot)$ at $(\bar{a}_4, \bar{b}_4)$ which is standard in the context of complementarity problems (see [21, Lemma 2.2]). The rest follows from (4.9). ∎

Remark 4.8. By comparing Propositions 4.3 and 4.7 we observe that for $b_4^* = 0$ the respective coderivative formulas coincide. This corresponds to the situation when we compute the values of $D^*\Phi(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}})(\boldsymbol{b}^*)$ for $(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}}) \in M_2$ as limits of values of $\widehat{D}^*\Phi(\boldsymbol{a}^{(i)}, \boldsymbol{b}^{(i)})(\boldsymbol{b}^{*(i)})$ with $(\boldsymbol{a}^{(i)}, \boldsymbol{b}^{(i)}) \in L$. Similarly if $b_4^* < 0$, then (4.10) amounts to the limit of the adjoint Jacobian of Φ in (4.5) when $a_4 \downarrow 0$. This corresponds to the situation when we approach $(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}}) \in M_2$ by a sequence $(\boldsymbol{a}^{(i)}, \boldsymbol{b}^{(i)}) \subset M_1$.

Consider now a point $(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}}) \in M_3^-$. By Table 4.1 there is an open interval $\mathcal{U}$ of $\bar{a}_4$ such that $\mathcal{U} \subset \mathrm{int}\mathbb{R}_+$ and

$$\Phi(\boldsymbol{a}) = \begin{bmatrix} \mathcal{F} a_4 \partial \|\boldsymbol{a}_{12}\| \\ 0 \\ 0 \end{bmatrix}$$

whenever $a_4 \in \mathcal{U}$.

Clearly, $\mathrm{Gr}\,\Phi$ locally around $(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}})$ coincides with the union $M_1 \cup M_3^+ \cup M_3^-$. Since these sets are disjoint, one has

$$N_{\mathrm{Gr}\,\Phi}(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}}) = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3, \tag{4.11}$$

where

$$\mathcal{N}_1 = \limsup_{(\boldsymbol{a},\boldsymbol{b}) \overset{M_1}{\to} (\bar{\boldsymbol{a}},\bar{\boldsymbol{b}})} \widehat{N}_{\mathrm{Gr}\,\Phi}(\boldsymbol{a}, \boldsymbol{b}),$$

$$\mathcal{N}_2 = \limsup_{(\boldsymbol{a},\boldsymbol{b}) \overset{M_3^+}{\to} (\bar{\boldsymbol{a}},\bar{\boldsymbol{b}})} \widehat{N}_{\mathrm{Gr}\,\Phi}(\boldsymbol{a}, \boldsymbol{b}),$$

and

$$\mathcal{N}_3 = \limsup_{(\boldsymbol{a},\boldsymbol{b}) \overset{M_3^-}{\to} (\bar{\boldsymbol{a}},\bar{\boldsymbol{b}})} \widehat{N}_{\mathrm{Gr}\,\Phi}(\boldsymbol{a}, \boldsymbol{b}).$$

By Proposition 4.5,

$$\mathcal{N}_2 = (\mathbb{R}^2 \times \{0\}_2) \times (\{0\}_2 \times \mathbb{R}^2).$$

The computation of $\mathcal{N}_1$ and $\mathcal{N}_3$ is, however, not so straightforward.

LEMMA 4.9. *Put* $\bar{\boldsymbol{w}} = \frac{\bar{\boldsymbol{b}}_{12}}{\mathcal{F}\bar{a}_4}$ *(so that* $\|\bar{\boldsymbol{w}}\| = 1$*). Then one has*

$$\begin{aligned}\mathcal{N}_1 = \{(\boldsymbol{a}^*, \boldsymbol{b}^*) \in \mathbb{R}^4 \times \mathbb{R}^4 | \boldsymbol{a}^*_{12} = 0,\, a^*_3 = 0,\, (a^*_4, \boldsymbol{b}^*_{12}) \in \mathbb{R}(-\mathcal{F}, \bar{\boldsymbol{w}})\} \\ \cup \{(\boldsymbol{a}^*, \boldsymbol{b}^*) \in \mathbb{R}^4 \times \mathbb{R}^4 | \boldsymbol{a}^*_{12} \in \bar{\boldsymbol{w}}^\perp,\, a^*_3 = a^*_4 = 0, \boldsymbol{b}^*_{12} = 0\}.\end{aligned} \tag{4.12}$$

Proof. We have to compute all accumulation points of sequences $(\boldsymbol{a}^{*(i)}, \boldsymbol{b}^{*(i)})$ satisfying the equations (see (4.5))

$$\begin{aligned}\boldsymbol{a}^{*(i)}_{12} &= -\frac{\mathcal{F}a^{(i)}_4}{\|\boldsymbol{a}^{(i)}_{12}\|}\begin{bmatrix} \frac{(a^{(i)}_2)^2}{\|\boldsymbol{a}^{(i)}_{12}\|^2} & \frac{-a^{(i)}_1 a^{(i)}_2}{\|\boldsymbol{a}^{(i)}_{12}\|^2} \\ \frac{-a^{(i)}_1 a^{(i)}_2}{\|\boldsymbol{a}^{(i)}_{12}\|^2} & \frac{(a^{(i)}_1)^2}{\|\boldsymbol{a}^{(i)}_{12}\|^2} \end{bmatrix}\begin{bmatrix} b^{*(i)}_1 \\ b^{*(i)}_2 \end{bmatrix}, \\ a^{*(i)}_3 &= 0, \\ a^{*(i)}_4 &= -\frac{\mathcal{F}}{\|\boldsymbol{a}^{(i)}_{12}\|}(a^i_1 b^{*(i)}_1 + a^i_2 b^{*(i)}_2)\end{aligned}$$

when $\|\boldsymbol{a}^{(i)}_{12}\| \downarrow 0$ and $a^{(i)}_4 \to \bar{a}_4 > 0$. Put $\boldsymbol{w}^{(i)} := \frac{\boldsymbol{a}^{(i)}_{12}}{\|\boldsymbol{a}^{(i)}_{12}\|}$ and

$$D^{(i)} := \begin{bmatrix} (w^{(i)}_2)^2 & -w^{(i)}_1 w^{(i)}_2 \\ -w^{(i)}_1 w^{(i)}_2 & (w^{(i)}_1)^2 \end{bmatrix}$$

so that $\boldsymbol{w}^{(i)} \to \bar{\boldsymbol{w}}$ and

$$D^{(i)} \to D := \begin{bmatrix} \bar{w}^2_2 & -\bar{w}_1\bar{w}_2 \\ -\bar{w}_1\bar{w}_2 & \bar{w}^2_1 \end{bmatrix}.$$

A sequence $\{(\boldsymbol{a}^{*(i)}_{12}, a^{*(i)}_4, \boldsymbol{b}^{*(i)}_{12})\}$ converges to an accumulation point $(\boldsymbol{a}^*_{12}, a^*_4, \boldsymbol{b}^*_{12})$ if and only if the sequence

$$\boldsymbol{\xi}^{(i)} := D^{(i)}\begin{bmatrix} \frac{b^{*(i)}_1}{\|\boldsymbol{a}^{(i)}_{12}\|} \\ \frac{b^{*(i)}_2}{\|\boldsymbol{a}^{(i)}_{12}\|} \end{bmatrix}$$

converges. This happens if either $\boldsymbol{b}^{*(i)}_{12} \in \mathrm{Ker} D (= \mathbb{R}\bar{\boldsymbol{w}})$ for all i, or $\boldsymbol{b}^*_{12} = 0$. The former case generates the first cone on the right-hand side of (4.12). In the latter case $\lim_{i\to\infty} \boldsymbol{\xi}^{(i)}$ can be everywhere in the range space of D which amounts to $\bar{\boldsymbol{w}}^\perp$. This gives rise to the second cone on the right-hand side of (4.12), and the statement has been established. ∎

LEMMA 4.10. *Consider a point* $(\boldsymbol{a}, \boldsymbol{b}) \in M_3^-$*, and put* $\boldsymbol{w} := \frac{\boldsymbol{b}_{12}}{\mathcal{F}a_4}$*. Then one has*

$$\widehat{N}_{\mathrm{Gr}\,\Phi}(\boldsymbol{a}, \boldsymbol{b}) = \{(\boldsymbol{a}^*, \boldsymbol{b}^*) \,|\, \langle \boldsymbol{a}^*_{12}, \boldsymbol{w}\rangle \le 0,\, a^*_3 = 0,\, (a^*_4, \boldsymbol{b}^*_{12}) \in \mathbb{R}_+(-\mathcal{F}, \boldsymbol{w})\}. \tag{4.13}$$

Proof. We compute first the contingent cone $T_{\mathrm{Gr}\,\Phi}(\boldsymbol{a}, \boldsymbol{b})$. To this end we observe that $\mathrm{Gr}\,\Phi$ locally around $(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}})$ coincides with the union $G_1 \cup G_2$, where

$$G_1 = \left\{(\boldsymbol{a}, \boldsymbol{b}) \,|\, \boldsymbol{a}_{12} \ne 0,\, \boldsymbol{b}_{12} = \mathcal{F}a_4\frac{\boldsymbol{a}_{12}}{\|\boldsymbol{a}_{12}\|},\, b_3 = b_4 = 0\right\}$$

and

$$G_2 = \{(\boldsymbol{a}, \boldsymbol{b}) \,|\, \boldsymbol{a}_{12} = 0,\, \boldsymbol{b}_{12} \in \mathcal{F} a_4 \mathbb{B},\, b_3 = b_4 = 0\}.$$

Clearly,

$$T_{\mathrm{Gr}\,\Phi}(\boldsymbol{a}, \boldsymbol{b}) = T_{G_1}(\boldsymbol{a}, \boldsymbol{b}) \cup T_{G_2}(\boldsymbol{a}, \boldsymbol{b}).$$

By the definition of the contingent cone,

$$T_{G_1}(\boldsymbol{a}, \boldsymbol{b}) = \Bigg\{ (\boldsymbol{h}, \boldsymbol{k}) | \exists \boldsymbol{h}^{(i)} \to \boldsymbol{h}, \boldsymbol{k}^{(i)} \to \boldsymbol{k}, \lambda^{(i)} \downarrow 0 \text{ such that}$$
$$\boldsymbol{b}_{12} + \lambda^{(i)} \boldsymbol{k}_{12}^{(i)} = \mathcal{F}(a_4 + \lambda^{(i)} h_4^{(i)}) \frac{\lambda^{(i)} \boldsymbol{h}_{12}^{(i)}}{\|\lambda^{(i)} \boldsymbol{h}_{12}^{(i)}\|},\, k_3^{(i)} = k_4^{(i)} = 0 \,\forall i \Bigg\}.$$

Then clearly

$$\frac{h_{12}^{(i)}}{\|h_{12}^{(i)}\|} \to w \quad \text{and} \quad k_{12}^{(i)} - \mathcal{F} h_4^{(i)} \frac{h_{12}^{(i)}}{\|h_{12}^{(i)}\|} = (\lambda^{(i)})^{-1} \mathcal{F} a_4 \left(\frac{h_{12}^{(i)}}{\|h_{12}^{(i)}\|} - w \right) \to w^{\perp}.$$

It follows that

$$\begin{aligned} T_{G_1}(\boldsymbol{a}, \boldsymbol{b}) &= \{(\boldsymbol{h}, \boldsymbol{k}) | \boldsymbol{h}_{12} \in \mathbb{R}_+ \boldsymbol{w}, \boldsymbol{k}_{12} - \mathcal{F} h_4 \boldsymbol{w} \in \boldsymbol{w}^{\perp},\, k_3 = k_4 = 0\} \\ &= \{(\boldsymbol{h}, \boldsymbol{k}) | \boldsymbol{h}_{12} \in \mathbb{R}_+ \boldsymbol{w}, \langle \boldsymbol{k}_{12}, \boldsymbol{w} \rangle - \mathcal{F} h_4 = 0, k_3 = k_4 = 0\}. \end{aligned}$$

In a similar way,

$$\begin{aligned} T_{G_2}(\boldsymbol{a}, \boldsymbol{b}) = \{(\boldsymbol{h}, \boldsymbol{k}) | \exists \boldsymbol{h}^{(i)} \to \boldsymbol{h}, \boldsymbol{k}^{(i)} \to \boldsymbol{k}, \lambda^{(i)} \downarrow 0 \text{ such that} \\ \boldsymbol{h}_{12}^{(i)} = 0, \boldsymbol{b}_{12} + \lambda^{(i)} \boldsymbol{k}_{12}^{(i)} \in \mathcal{F}(a_4 + \lambda^{(i)} h_4^{(i)}) \mathbb{B},\, k_3^{(i)} = k_4^{(i)} = 0 \,\forall i\} \end{aligned}$$

so that

$$\begin{aligned} T_{G_2}(\boldsymbol{a}, \boldsymbol{b}) &= \{(\boldsymbol{h}, \boldsymbol{k}) | \boldsymbol{h}_{12} = 0, \langle \boldsymbol{k}_{12} - \mathcal{F} h_4 \boldsymbol{w}, \boldsymbol{w} \rangle \le 0, k_3 = k_4 = 0\} \\ &= \{(\boldsymbol{h}, \boldsymbol{k}) | \boldsymbol{h}_{12} = 0, \langle \boldsymbol{k}_{12}, \boldsymbol{w} \rangle - \mathcal{F} h_4 \le 0, k_3 = k_4 = 0\}. \end{aligned}$$

By computing the negative polars we obtain that

$$\widehat{N}_{G_1}(\boldsymbol{a}, \boldsymbol{b}) = \{(\boldsymbol{a}^*, \boldsymbol{b}^*) | \langle \boldsymbol{a}_{12}^*, \boldsymbol{w} \rangle \le 0, a_3^* = 0,\, (a_4^*, \boldsymbol{b}_{12}^*) \in \mathbb{R}(-\mathcal{F}, \boldsymbol{w})\}$$

and

$$\widehat{N}_{G_2}(\boldsymbol{a}, \boldsymbol{b}) = \{(\boldsymbol{a}^*, \boldsymbol{b}^*) | a_3^* = 0,\, (a_4^*, \boldsymbol{b}_{12}^*) \in \mathbb{R}_+(-\mathcal{F}, \boldsymbol{w})\}.$$

In this way we infer that

$$\begin{aligned} \widehat{N}_{\mathrm{Gr}\,\Phi}(\boldsymbol{a}, \boldsymbol{b}) &= \widehat{N}_{G_1}(\boldsymbol{a}, \boldsymbol{b}) \cap \widehat{N}_{G_2}(\boldsymbol{a}, \boldsymbol{b}) \\ &= \{(\boldsymbol{a}^*, \boldsymbol{b}^*) | \langle \boldsymbol{a}_{12}^*, \boldsymbol{w} \rangle \le 0, a_3^* = 0,\, (a_4^*, \boldsymbol{b}_{12}^*) \in \mathbb{R}_+(-\mathcal{F}, \boldsymbol{w})\}, \end{aligned}$$

and we are done. □

On the basis of Lemmas 4.9 and 4.10 we arrive now at the following exact description of $D^*\Phi$ in the case of M_3^-.

THEOREM 4.11. *Let* $(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}}) \in M_3^-$ *and* $\bar{\boldsymbol{w}} = \frac{\bar{\boldsymbol{b}}_{12}}{\mathcal{F}\bar{a}_4}$. *Then one has*

(4.14)
$$D^*\Phi(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}})(\boldsymbol{b}^*) = \begin{cases} \{\boldsymbol{a}^* \in \mathbb{R}^4 | \boldsymbol{a}^*_{12} = 0, a^*_3 = 0, a^*_4 = \mathcal{F}\alpha\} & \text{if } \boldsymbol{b}^*_{12} = \alpha\bar{\boldsymbol{w}}, \alpha \geq 0, \\ \{\boldsymbol{a}^* \in \mathbb{R}^4 | \langle \boldsymbol{a}^*_{12}, \bar{\boldsymbol{w}}\rangle \leq 0, a^*_3 = 0, a^*_4 = \mathcal{F}\alpha\} & \text{if } \boldsymbol{b}^*_{12} = \alpha\bar{\boldsymbol{w}}, \alpha \leq 0, \\ \mathbb{R}^2 \times \{0\}_2 & \text{if } \boldsymbol{b}^*_{12} = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. On the basis of Lemma 4.10 we can compute the cone $\mathcal{N}_3$. To this end it suffices to realize that $a_4 \to \bar{a}_4 (> 0)$ and $\boldsymbol{b}_{12} \to \bar{\boldsymbol{b}}_{12}$ implies $\boldsymbol{w} = \frac{\boldsymbol{b}_{12}}{\mathcal{F}a_4} \to \bar{\boldsymbol{w}} = \frac{\bar{\boldsymbol{b}}_{12}}{\mathcal{F}\bar{a}_4}$. Consequently, $\mathcal{N}_3 = \widehat{N}_{\operatorname{Gr}\Phi}(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}})$. The rest follows from (4.11) and the definition of the coderivative. ∎

In the case of M_4, however, this approach would be substantially more difficult. The respective limiting normal cone will again have the structure of a union, some parts of which can easily be computed. The set M_4 consists, however, only of one point (when we ignore variable a_3), and so the absence of an exact formula does not have a negative impact on the proposed numerical method. From this reason we also omit a detailed analysis of this case here.

All derived formulas (4.4), (4.5), (4.6), (4.10), and (4.14) enable us to specify for each node i a linear subspace

$$\mathcal{L}_i \in \operatorname{Gr} D^*\Phi(\bar{\boldsymbol{y}}^i, -F^i(\bar{\boldsymbol{\alpha}})\bar{\boldsymbol{y}} + l^i(\bar{\boldsymbol{\alpha}}))$$

such that the AGE (4.3) amounts to the linear system

$$\begin{aligned} 0 &= \boldsymbol{y}^* + (F(\bar{\boldsymbol{\alpha}}))^T \boldsymbol{v} + \boldsymbol{w} \\ &(w^i, v^i) \in \mathcal{L}_i, \quad i = 1, 2, \dots p. \end{aligned} \tag{4.15}$$

This of course facilitates its numerical solution.

As already mentioned, if some nodes lie in M_2 or M_3^- (or in M_4), this technique need not immediately lead to a correct subgradient due to the right inclusion in (4.1). Fortunately, as shown by the performed numerical tests, such nodes occur very rarely during the iteration process and typically cause some difficulties only close to optimal points. Formulas (4.10) and (4.14) offer then a possibility to restore the convergence by a suitable change of the adjoint system (4.15).

5. Numerical results. The results of the previous sections will now be used for construction of a numerical method for the solution of $(\tilde{\mathbb{P}})$. We assume that the friction coefficient $\mathcal{F}$ is small enough in the sense of Theorem 3.8 so that the solution of the contact problem with Coulomb friction is unique. Further, as in section 4, we assume that the cost functional J is continuously differentiable. For the minimization of Θ we use the BT code [27] based on the bundle-truss algorithm of Schramm and Zowe [25]. In every step of the iteration process, this code has to be supplied with the function value $\Theta(\boldsymbol{\alpha})$ and one (arbitrary) Clarke's subgradient of Θ at $\boldsymbol{\alpha}$.

5.1. Solving the state problem. To compute a function value $J(\boldsymbol{\alpha}, S(\boldsymbol{\alpha}))$, we have to evaluate $S(\boldsymbol{\alpha})$, i.e., to solve the fixed point problem $(\mathcal{P})$. For that, we use the method of successive approximations introduced in Corollary 3.9. Each iterative step requires us to solve the contact problem with given friction $(\mathcal{M}(\boldsymbol{L}, \boldsymbol{g}))$, in which the slip bound $\boldsymbol{g}$ is updated by the result of the previous iteration, i.e., $\boldsymbol{g} \equiv \boldsymbol{\lambda}^{(k)}$. The problem $(\mathcal{M}(\boldsymbol{L}, \boldsymbol{g}))$ is solved using the so-called *reciprocal variational formulation* (see [8, 11, 12]). As in the previous sections, we denote by $\boldsymbol{\lambda} \in \mathbb{R}^p$ the vector of normal

contact stresses. Further, let $\boldsymbol{\tau} \in \mathbb{R}^{2p}$ be the vector of tangential contact stresses. Notice that with each contact node we associate one component λ_i of $\boldsymbol{\lambda}$ and two components τ_{2i-1}, τ_{2i} of $\boldsymbol{\tau}$. After eliminating the first variable $\boldsymbol{u}$ from $(\mathcal{M}(\boldsymbol{L}, \boldsymbol{g}))$, we arrive at the problem formulated in terms of contact stresses:

$$\min_{\boldsymbol{\sigma} \in \mathbb{S}} f(\boldsymbol{\sigma}) := \frac{1}{2}\boldsymbol{\sigma}^T \boldsymbol{Q}\boldsymbol{\sigma} - \boldsymbol{\sigma}^T \boldsymbol{h} \tag{5.1}$$

with

$$\mathbb{S} = \{\boldsymbol{\sigma} = (\boldsymbol{\lambda}, \boldsymbol{\tau}) \in \mathbb{R}^{3p} \mid \lambda_i \geq 0,\ \tau_{2i-1}^2 + \tau_{2i}^2 \leq g_i^2, \quad i = 1, 2, \ldots, p\},$$

where $\boldsymbol{Q} = \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}^T$, $\boldsymbol{h} = \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{L} + \boldsymbol{c}$, $\boldsymbol{B} = (\boldsymbol{N}^T, \boldsymbol{T}^T)^T$, $\boldsymbol{c} = (\boldsymbol{\alpha}^T, 0^T)^T$, and $\boldsymbol{T} \in \mathbb{R}^{2p \times n}$ stands for a matrix representation of the linear mapping T used in (3.3). After computing $\boldsymbol{\lambda}, \boldsymbol{\tau}$ from (5.1), one obtains the eliminated variable $\boldsymbol{u}$ by

$$\boldsymbol{u} = \boldsymbol{A}^{-1}(\boldsymbol{L} - \boldsymbol{N}\boldsymbol{\lambda} - \boldsymbol{T}\boldsymbol{\tau}).$$

As (5.1) is a strictly convex problem with quadratic objective and separable quadratic constraints, it can be solved by the algorithm proposed by Kučera in [14] and analyzed in [15]. The algorithm generalizes ideas of Dostál and Schöberl [5] originally proposed for convex quadratic problems with simple bounds. Because an efficient solution procedure for (5.1) is essential for the overall efficiency of our numerical approach, we give a brief description of the algorithm.

Let $\mathcal{N} = \{1, \ldots, 3p\}$ be the set of all indices. At a point $\boldsymbol{\sigma} \in \mathbb{S}$, we denote the gradient of f by $\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{\sigma}) = \boldsymbol{Q}\boldsymbol{\sigma} - \boldsymbol{h}$ and introduce an active set $\mathcal{A} \subseteq \mathcal{N}$ by

$$\mathcal{A} := \mathcal{A}(\boldsymbol{\sigma}) = \{i \mid \lambda_i = 0\} \cup \{j \mid j = 2i - 1 + p : \tau_{2i-1}^2 + \tau_{2i}^2 = g_i^2\} \cup \{j \mid j = 2i + p : \tau_{2i-1}^2 + \tau_{2i}^2 = g_i^2\}.$$

Using the orthogonal projection $P_{\mathbb{S}} : \mathbb{R}^{3p} \mapsto \mathbb{S}$, we define the *projected gradient* for a fixed $\widetilde{\alpha} \geq 0$ as

$$\widetilde{\boldsymbol{r}} = \widetilde{\boldsymbol{r}}(\boldsymbol{\sigma}) = \frac{1}{\widetilde{\alpha}}(\boldsymbol{\sigma} - P_{\mathbb{S}}(\boldsymbol{\sigma} - \widetilde{\alpha}\boldsymbol{r}(\boldsymbol{\sigma}))).$$

Notice that the projected gradient enables us to write down the optimality criterion characterizing the solution $\boldsymbol{\sigma}^*$ of (5.1) in the form $\widetilde{\boldsymbol{r}}(\boldsymbol{\sigma}^*) = 0$. Our algorithm is based on the observation that nonzero components of $\widetilde{\boldsymbol{r}}(\boldsymbol{\sigma})$ at $\boldsymbol{\sigma} \neq \boldsymbol{\sigma}^*$ determine descent directions changing appropriately the active set. To this end, we introduce components of $\widetilde{\boldsymbol{r}}(\boldsymbol{\sigma})$ and $\boldsymbol{r}(\boldsymbol{\sigma})$ called the *projected free gradient* $\widetilde{\boldsymbol{\varphi}} = \widetilde{\boldsymbol{\varphi}}(\boldsymbol{\sigma})$, the *projected boundary gradient* $\widetilde{\boldsymbol{\beta}} = \widetilde{\boldsymbol{\beta}}(\boldsymbol{\sigma})$, and the *free gradient* $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\boldsymbol{\sigma})$, respectively, defined by

$$\begin{aligned} \widetilde{\boldsymbol{\varphi}}_{\mathcal{A}} &= 0, & \widetilde{\boldsymbol{\varphi}}_{\mathcal{N}\setminus\mathcal{A}} &= \widetilde{\boldsymbol{r}}_{\mathcal{N}\setminus\mathcal{A}}, \\ \widetilde{\boldsymbol{\beta}}_{\mathcal{A}} &= \widetilde{\boldsymbol{r}}_{\mathcal{A}}, & \widetilde{\boldsymbol{\beta}}_{\mathcal{N}\setminus\mathcal{A}} &= 0, \\ \boldsymbol{\varphi}_{\mathcal{A}} &= 0, & \boldsymbol{\varphi}_{\mathcal{N}\setminus\mathcal{A}} &= \boldsymbol{r}_{\mathcal{N}\setminus\mathcal{A}}. \end{aligned}$$

We combine three steps to generate a sequence of iterates $\{\boldsymbol{\sigma}^{(l)}\}$ that approximates the solution to (5.1):

- the *expansion step*: $\boldsymbol{\sigma}^{(l+1)} = \boldsymbol{\sigma}^{(l)} - \widetilde{\alpha}\widetilde{\boldsymbol{\varphi}}(\boldsymbol{\sigma}^{(l)})$,
- the *proportioning step*: $\boldsymbol{\sigma}^{(l+1)} = \boldsymbol{\sigma}^{(l)} - \widetilde{\alpha}\widetilde{\boldsymbol{\beta}}(\boldsymbol{\sigma}^{(l)})$,

- the *conjugate gradient step*: $\boldsymbol{\sigma}^{(l+1)} = \boldsymbol{\sigma}^{(l)} - \alpha_{cg}^{(l)} \boldsymbol{p}^{(l)}$, where the step size $\alpha_{cg}^{(l)}$ and the conjugate gradient directions $\boldsymbol{p}^{(l)}$ are computed recurrently (see [7]) so that the recurrence starts from $\boldsymbol{\sigma}^{(s)}$ generated by the last expansion or proportioning step and $\mathcal{A}(\boldsymbol{\sigma}^{(l+1)}) = \mathcal{A}(\boldsymbol{\sigma}^{(s)})$.

The expansion step may add indices, while the proportioning step may release indices to/from the current active set. The conjugate gradient step is used to carry out efficiently minimization of the objective f on the interior of the set $W(\boldsymbol{\sigma}^{(s)}) = \{\boldsymbol{\sigma} \in \mathbb{S} \mid \boldsymbol{\sigma}_{\mathcal{A}} = \boldsymbol{\sigma}_{\mathcal{A}}^{(s)}, \mathcal{A} = \mathcal{A}(\boldsymbol{\sigma}^{(s)})\}$. Moreover, the algorithm exploits a given constant $\widetilde{\Gamma} > 0$ and the *releasing criterion*

$$\widetilde{\boldsymbol{\beta}}(\boldsymbol{\sigma}^{(l)})^T \boldsymbol{r}(\boldsymbol{\sigma}^{(l)}) \le \widetilde{\Gamma}\ \widetilde{\boldsymbol{\varphi}}(\boldsymbol{\sigma}^{(l)})^T \boldsymbol{r}(\boldsymbol{\sigma}^{(l)}) \tag{5.2}$$

to decide which of the steps will be performed.

ALGORITHM 1. *Let $\boldsymbol{\sigma}^{(0)} \in \mathbb{S}$, $\widetilde{\Gamma} > 0$, $\widetilde{\alpha} \in (0, \|\boldsymbol{Q}\|^{-1}]$, and $\varepsilon \ge 0$ be given. For $\boldsymbol{\sigma}^{(l)}$, $\boldsymbol{\sigma}^{(s)}$ known, $0 \le s \le l$, where $\boldsymbol{\sigma}^{(s)}$ is computed by the last expansion or proportioning step, choose $\boldsymbol{\sigma}^{(l+1)}$ by the following rules:*

(i) *If $\|\widetilde{\boldsymbol{r}}(\boldsymbol{\sigma}^{(l)})\| \le \varepsilon$, return $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(l)}$.*

(ii) *If $\boldsymbol{\sigma}^{(l)}$ fulfills* (5.2)*, try to generate $\boldsymbol{\sigma}^{(l+1)}$ by the conjugate gradient step. If $\boldsymbol{\sigma}^{(l+1)} \in \operatorname{Int} W(\boldsymbol{\sigma}^{(s)})$, accept it, else generate $\boldsymbol{\sigma}^{(l+1)}$ by the expansion step.*

(iii) *If $\boldsymbol{\sigma}^{(l)}$ does not fulfil* (5.2)*, generate $\boldsymbol{\sigma}^{(l+1)}$ by the proportioning step.*

Contrary to the simple bound problem analyzed in [5], the algorithm does not exhibit the finite terminating property; the same convergence rate is, however, achieved. In [15] one can find the following statement.

THEOREM 5.1. *Let $\boldsymbol{\sigma}^* \in \mathbb{S}$ denote the solution to* (5.1)*, α_{min} denote the smallest eigenvalue of $\boldsymbol{Q}$, and $\widehat{\Gamma} = \max\{\widetilde{\Gamma}, \widetilde{\Gamma}^{-1}\}$. Let $\{\boldsymbol{\sigma}^{(l)}\}$ be the sequence generated by Algorithm 1 with $\varepsilon = 0$. Then*

$$f(\boldsymbol{\sigma}^{(l+1)}) - f(\boldsymbol{\sigma}^*) \le \eta \left(f(\boldsymbol{\sigma}^{(l)}) - f(\boldsymbol{\sigma}^*) \right),$$

where

$$\eta = 1 - \frac{\widetilde{\alpha}\alpha_{min}}{2 + 2\widehat{\Gamma}^2} < 1.$$

The error in the $\boldsymbol{Q}$-energy norm is bounded by

$$\|\boldsymbol{\sigma}^{(l)} - \boldsymbol{\sigma}^*\|_Q^2 \le 2\eta^l \left(f(\boldsymbol{\sigma}^{(0)}) - f(\boldsymbol{\sigma}^*) \right).$$

Theorem 5.1 yields the best value of the convergence rate factor η for the choice $\widetilde{\Gamma} = \hat{\Gamma} = 1$ and $\widetilde{\alpha} = \|\boldsymbol{Q}\|^{-1}$. Then

$$\eta = 1 - \frac{1}{4}\kappa(\boldsymbol{Q}),$$

where $\kappa(\boldsymbol{Q})$ is the spectral condition number of $\boldsymbol{Q}$.

5.2. Numerical examples. In order to work with a relatively small number of control variables and, at the same time, to get a smooth shape of the optimal boundary, we will model the contact boundary Γ_c by a Bézier surface of order d. The design variable $\boldsymbol{\alpha}$ is thus the vector of its control points. The Bézier surface ϑ_α of order (d_1, d_2) in $\overline{\mathcal{R}}(= [0, a] \times [0, b])$ is generated by a $d_1 \times d_2$ matrix $\boldsymbol{\alpha}$ as

$$\vartheta_\alpha(x_1, x_2) = \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} \alpha^{(i,j)} \beta_{d_1}^i(x_1) \beta_{d_2}^j(x_2), \quad (x_1, x_2) \in \overline{\mathcal{R}},$$

where

$$\beta_{d_1}^{\ell}(x_1) = \frac{1}{a^{d_1}} \binom{d_1}{\ell} x_1^{\ell}(a - x_1)^{d_1-\ell}, \quad \beta_{d_2}^{\ell}(x_2) = \frac{1}{b^{d_2}} \binom{d_2}{\ell} x_2^{\ell}(b - x_2)^{d_2-\ell}.$$

The corner points of ϑ_α are identical to the "corner elements" of the control matrix. The surface itself lies in the convex hull of the control points. This means that any upper and lower bounds on the control points hold for the whole surface in $\overline{\mathcal{R}}$, too.

The discrete shape optimization problem is now defined as follows:

$$(\tilde{\mathbb{P}}_{\mathcal{B}}) \qquad \left.\begin{array}{l} \text{minimize } J(\boldsymbol{\alpha}, S(\boldsymbol{\alpha})) \\ \text{subject to } \boldsymbol{\alpha} \in \mathcal{U}, \end{array}\right\}$$

where $\mathcal{U}$ is given by

$$\begin{aligned} \mathcal{U} = \Bigg\{ & \boldsymbol{\alpha} \in \mathbb{R}^{d_1 \times d_2} \mid 0 \le \alpha^{(i,j)} \le C_0,\ i = 0, 1, \dots, d_1,\ j = 0, 1, \dots, d_2; \\ & |\alpha^{(i+1,j)} - \alpha^{(i,j)}| \le C_1 \frac{a}{d_1},\ i = 0, 1, \dots, d_1 - 1,\ j = 0, 1, \dots, d_2; \\ & |\alpha^{(i,j+1)} - \alpha^{(i,j)}| \le C_1 \frac{b}{d_2},\ i = 0, 1, \dots, d_1,\ j = 0, 1, \dots, d_2 - 1; \\ & \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} \alpha^{(i,j)} = C_2(d_1 + 1)(d_2 + 1) \Bigg\}, \end{aligned}$$

where C_0, C_1, and C_2 are given positive constants. The first set of constraints guarantees that $|F_\alpha(x)| \le C_0$ for all $x \in \overline{\mathcal{R}}$. The second and third constraint sets take care of the slopes of ϑ_α in the direction of axes x_1, x_2. It is well known that if the control points satisfy these conditions, then $|\frac{\partial}{\partial x_k} \vartheta_\alpha(x_1, x_2)| \le C_1$ for all $(x_1, x_2) \in \overline{\mathcal{R}}$, $k = 1, 2$.

The equality constraint is added to control the volume of the domain by the control points of the Bézier surface. The number $(c - C_2)ab$ equals the volume of $\Omega(\boldsymbol{\alpha})$ defined by

$$\Omega(\boldsymbol{\alpha}) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1, x_2) \in (0, a) \times (0, b),\ F_\alpha(x_1, x_2) < x_3 < c\}; \tag{5.3}$$

see Figure 5.1 which shows the body in 3D and 2D view. Thus, the equality constraint has a physical meaning of preserving the weight of the structure.

All test examples solved below differ only in the cost function. The shape of the elastic body $\Omega(\boldsymbol{\alpha}), \boldsymbol{\alpha} \in \mathcal{U}$, is given by (5.3), with $a = 2$, $b = 1$, and $c = 1$. The set of admissible designs $\mathcal{U}$ is determined by the choice $C_0 = 0.75$, $C_1 = 0.5$, and $C_2 = 0.01$.

The left-hand face $\Gamma_u = \{x \in \Omega(\boldsymbol{\alpha}) \mid x_1 = 0\}$ is the part of the boundary with the prescribed Dirichlet condition where all displacements are fixed to zero. The nonzero external loads are defined as follows. The top face $\Gamma_{P1} = \{x \in \Omega(\boldsymbol{\alpha}) \mid x_3 = 1\}$ is subjected to constant pressure $P_1 = -8 \cdot 10^{-2} \frac{\mathrm{N}}{\mathrm{m}^2}$. The right-hand face $\Gamma_{P2} = \{x \in \Omega(\boldsymbol{\alpha}) \mid x_1 = 1\}$ is subjected to constant traction $P_2 = 5 \cdot 10^{-2} \frac{\mathrm{N}}{\mathrm{m}^2}$. The bottom face Γ_c represented by the graph of $\vartheta(\boldsymbol{\alpha})$ is supported by a rigid half-space $\mathbb{R}^2 \times \mathbb{R}_-$.

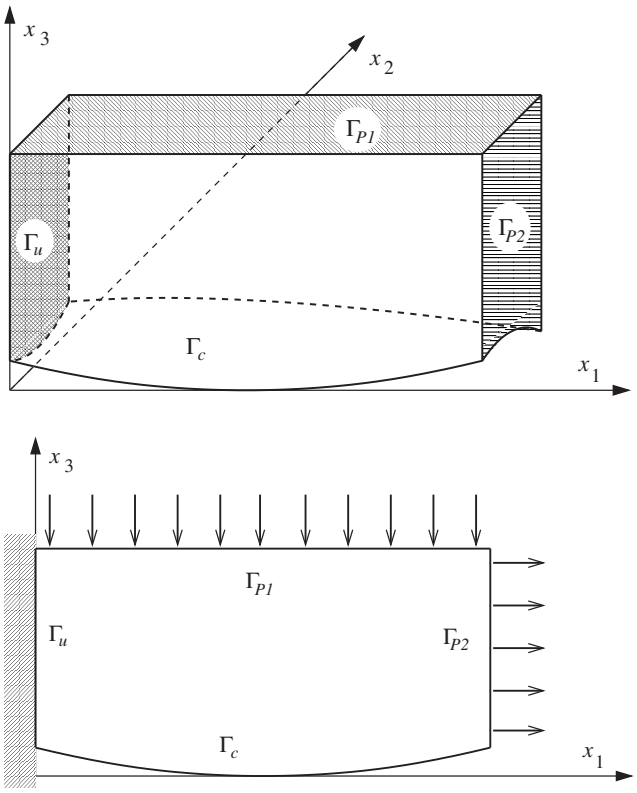

FIG. 5.1. *The elastic body and applied loads.*

The examples were solved with the Young modulus $E = 21.19$ Pa, Poisson's constant $\sigma = 0.277$, and the friction coefficient $\mathcal{F} = 0.3$.

In Example 1 the reference body, a prism of size $2 \times 1 \times 1$, was uniformly carved into $32 \times 15 \times 9 = 4320$ bricks. The finite element discretization was constructed by using trilinear elements. The total number of nodal displacements was 15360 including 1536 contact displacements. In Examples 2 and 3 we have used a coarser discretization of $24 \times 11 \times 11 = 2904$ bricks, giving 10398 nodal degrees of freedom and 864 contact displacements. The partition of each $\Omega(\boldsymbol{\alpha})$ was constructed from the partition of the reference body $\hat{\Omega}$ by a suitable coordinate transformation in the x_3 direction. The total number of design variables (control points of the Bézier surface) was 32 ($d_1 = 8, d_2 = 4$).

Example 1. We try to find a shape of the contact surface for which the normal stress is as close as possible to a prescribed function. The corresponding problem ($\tilde{\mathbb{P}}$) can be formulated as

$$\begin{aligned} &\text{minimize } \|\overline{\boldsymbol{\lambda}}_\nu - \boldsymbol{\lambda}_\nu\|_2^2 \\ &\text{subject to} \quad \boldsymbol{\alpha} \in \mathcal{U}, \end{aligned}$$

where $\overline{\boldsymbol{\lambda}}_\nu$ is the vector of desired normal stresses. This target function is a step function, depicted in Figure 5.2 (left). Figure 5.2 (right) shows the distribution of the contact normal stresses for the initial shape, given by the constant vector $\boldsymbol{\alpha}_0 = [0.01, \ldots, 0.01]$.

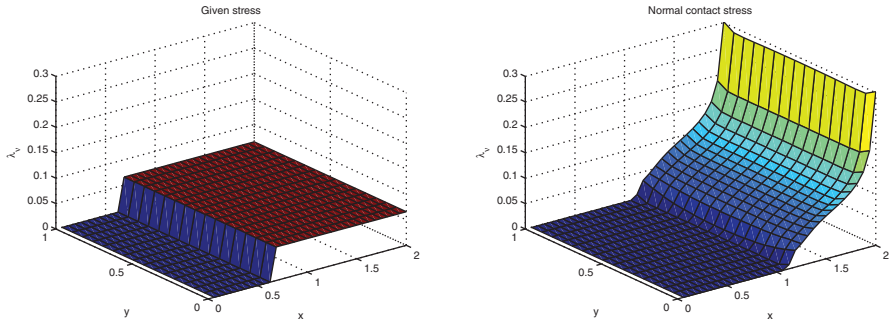

FIG. 5.2. *Example* 1, *target normal stress (left) and normal stress for initial design (right).*

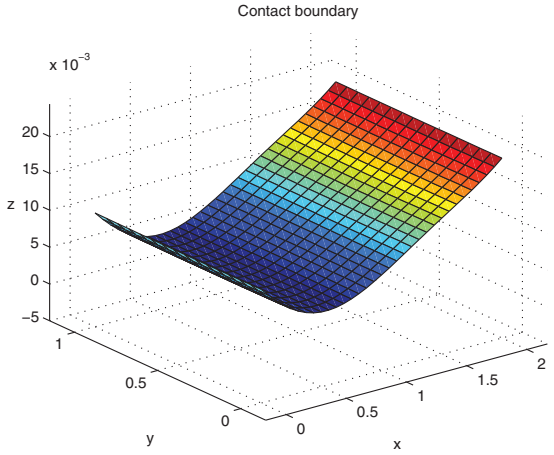

FIG. 5.3. *Example* 1, *optimal design.*

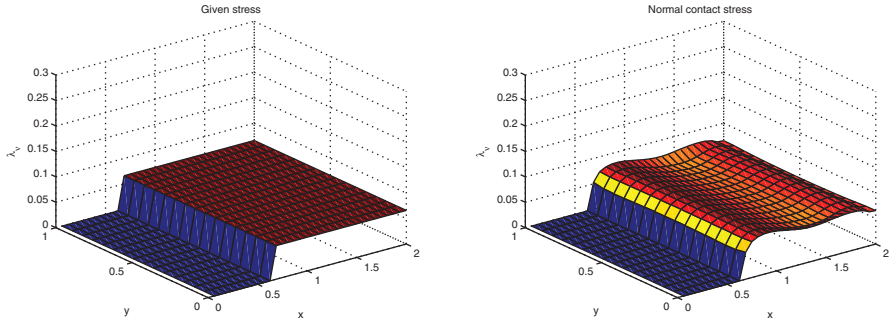

FIG. 5.4. *Example* 1, *target normal stress (left) and normal stress for optimal design (right).*

The objective function value for the initial design was equal to $J(\boldsymbol{\alpha}_0) = 1.9073606 \cdot 10^{-5}$. The stopping parameter for the code BT was set to $\varepsilon = 1 \cdot 10^{-4}$. This required precision was reached after 307 iterations. (With a stopping tolerance decreased to $\varepsilon = 1 \cdot 10^{-3}$, the code finished already after 12 iterations.) Figure 5.3 presents the optimal solution, i.e., the optimal shape of the contact boundary, while Figure 5.4 compares the contact normal stresses with the prescribed values. We see that the stresses for the optimal shape follow the step function rather closely.

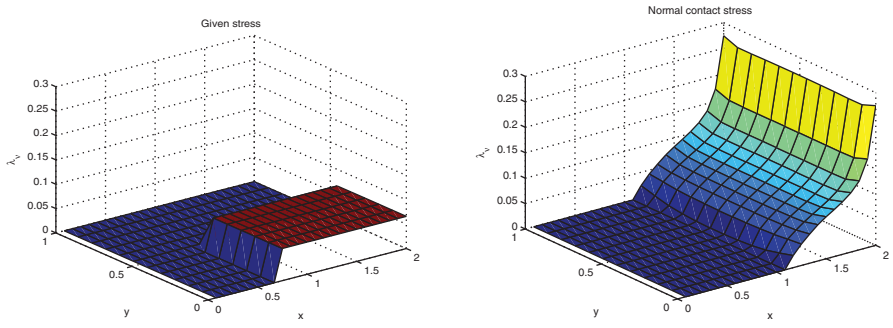

FIG. 5.5. *Example* 2, *target normal stress (left) and normal stress for initial design (right).*

Finally, we present the optimal shape in terms of the matrix of the respective control points:

$$\boldsymbol{\alpha}_{\text{opt}} = \begin{bmatrix} 0.0104 & 0.0104 & 0.0104 & 0.0104 \\ 0.0052 & 0.0050 & 0.0050 & 0.0052 \\ 0.0016 & 0.0016 & 0.0016 & 0.0016 \\ 0.0048 & 0.0052 & 0.0052 & 0.0048 \\ 0.0107 & 0.0109 & 0.0109 & 0.0107 \\ 0.0129 & 0.0124 & 0.0124 & 0.0129 \\ 0.0151 & 0.0150 & 0.0150 & 0.0151 \\ 0.0194 & 0.0194 & 0.0194 & 0.0194 \end{bmatrix}.$$

The optimal value of the objective function was $J_{\text{opt}} = 2.8432364 \cdot 10^{-7}$.

Example 2. We consider the same objective function as in the previous example, but this time we want to identify a "nonsymmetric" prescribed function shown in Figure 5.5 (left). Figure 5.5 (right) shows the distribution of the contact normal stresses for the initial shape, again given by the constant vector $\boldsymbol{\alpha}_0 = [0.01, \ldots, 0.01]$.

The objective value for the initial design was equal to $J(\boldsymbol{\alpha}_0) = 4.635256 \cdot 10^{-5}$. For the stopping criterion $\varepsilon = 1 \cdot 10^{-4}$ we needed 233 iterations of the code BT. (With a stopping tolerance decreased to $\varepsilon = 1 \cdot 10^{-3}$, the code finished already after 20 iterations.) Figure 5.6 presents the optimal solution, i.e., the optimal shape of the contact boundary, while Figure 5.7 compares the contact normal stresses with the prescribed values.

Finally, we present the optimal shape in terms of the matrix of the respective control points:

$$\boldsymbol{\alpha}_{\text{opt}} = \begin{bmatrix} 0.0071 & 0.0069 & 0.0066 & 0.0067 \\ 0.0048 & 0.0045 & 0.0046 & 0.0053 \\ 0.0019 & 0.0022 & 0.0045 & 0.0053 \\ 0.0034 & 0.0044 & 0.0094 & 0.0096 \\ 0.0075 & 0.0085 & 0.0147 & 0.0149 \\ 0.0085 & 0.0092 & 0.0148 & 0.0160 \\ 0.0112 & 0.0133 & 0.0185 & 0.0205 \\ 0.0144 & 0.0169 & 0.0212 & 0.0228 \end{bmatrix}.$$

The optimal value of the objective function was $J_{\text{opt}} = 1.2315453 \cdot 10^{-5}$.

Example 3. Finally, we solve the same problem as in the previous two examples, now with a prescribed stress distribution as shown in Figure 5.8 (left). Figure 5.8

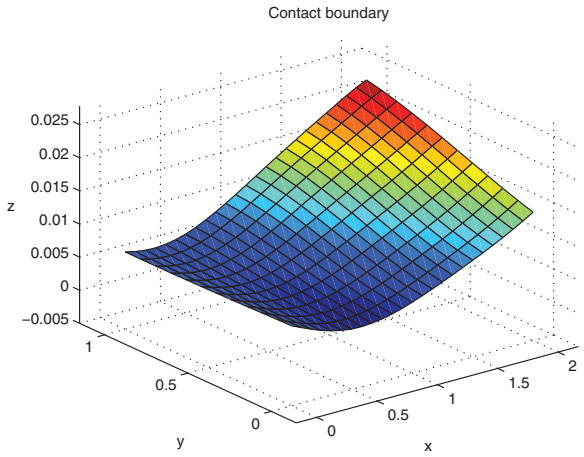

FIG. 5.6. *Example* 2*, optimal design.*

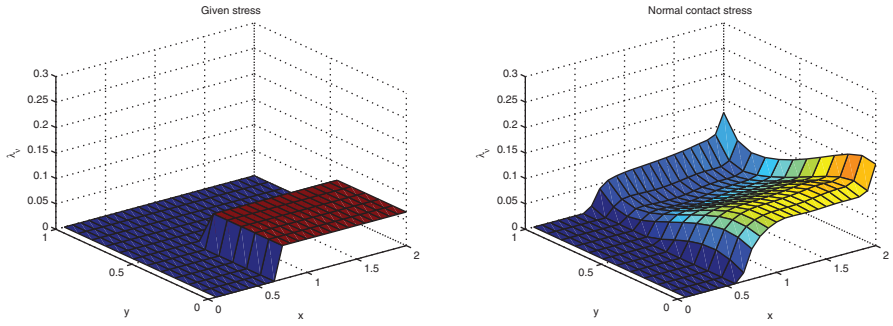

FIG. 5.7. *Example* 2*, target normal stress (left) and normal stress for optimal design (right).*

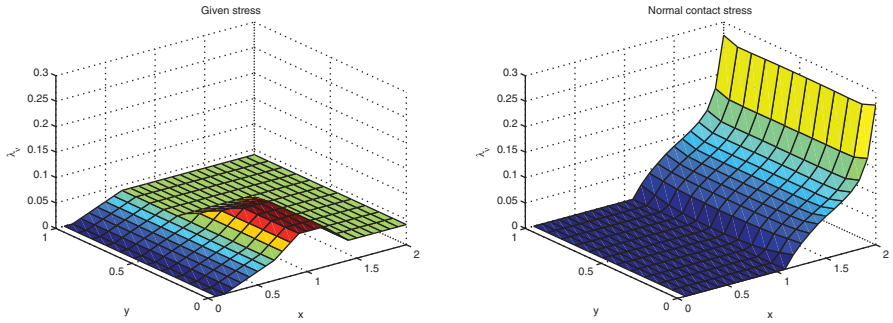

FIG. 5.8. *Example* 3*, target normal stress (left) and normal stress for initial design (right).*

(right) shows the distribution of the contact normal stresses for the initial shape given, as before, by the constant vector $\boldsymbol{\alpha}_0 = [0.01, \ldots, 0.01]$.

The objective value for the initial design was equal to $J(\boldsymbol{\alpha}_0) = 4.3084971 \cdot 10^{-5}$. To reach the precision $\varepsilon = 1 \cdot 10^{-4}$ the BT code required 286 iterations. (With a stopping tolerance decreased to $\varepsilon = 1 \cdot 10^{-3}$, the code needed only 24 iterations.) Figure 5.9 shows the optimal shape of the contact boundary, while Figure 5.10 compares the contact normal stresses with the prescribed values.

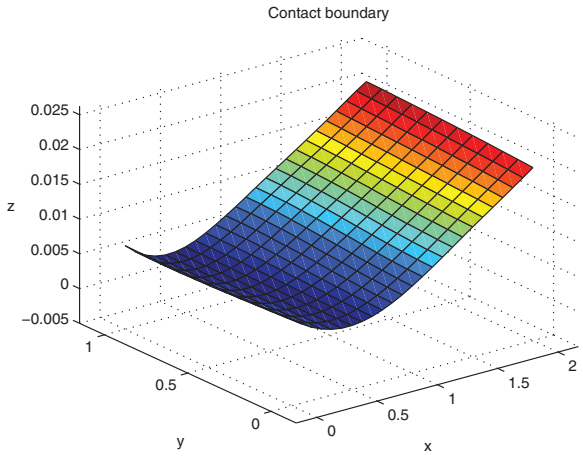

FIG. 5.9. *Example* 3*, optimal design.*

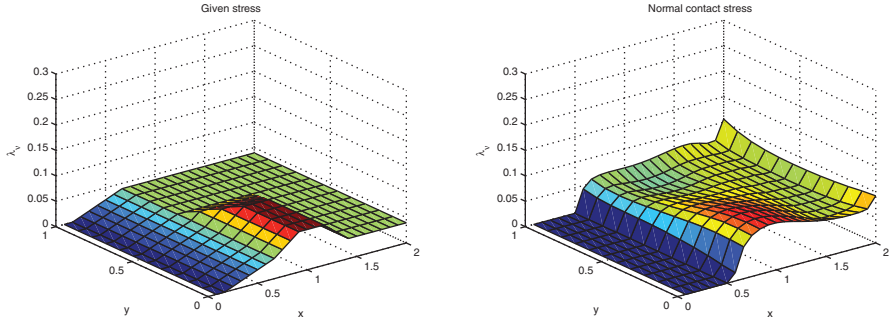

FIG. 5.10. *Example* 3*, target normal stress (left) and normal stress for optimal design (right).*

The optimal shape in terms of the matrix of the respective control points is given below:

$$\boldsymbol{\alpha}_{\text{opt}} = \begin{bmatrix} 0.0074 & 0.0071 & 0.0067 & 0.0070 \\ 0.0034 & 0.0028 & 0.0018 & 0.0022 \\ 0.0022 & 0.0024 & 0.0022 & 0.0021 \\ 0.0045 & 0.0055 & 0.0083 & 0.0078 \\ 0.0092 & 0.0091 & 0.0125 & 0.0123 \\ 0.0134 & 0.0129 & 0.0129 & 0.0131 \\ 0.0160 & 0.0173 & 0.0172 & 0.0181 \\ 0.0199 & 0.0205 & 0.0210 & 0.0212 \end{bmatrix}.$$

The optimal value of the objective function was $J_{\text{opt}} = 4.1426132 \cdot 10^{-6}$.

Acknowledgments. The authors would like to thank the anonymous referees and the associate editor for their constructive comments.

REFERENCES

[1] R. L. Benedict and J. E. Taylor, *Optimal design for elastic bodies in contact*, in Optimization of Distributed Parameter Structures, Part II, E. J. Haug and J. Céa, eds., Sijthoff & Noorddhoff, Alphen aan den Rijn, The Netherlands, 1981, pp. 1553–1599.

[2] P. Beremlijski, J. Haslinger, M. Kočvara, and J. Outrata, *Shape optimization in contact problems with Coulomb friction*, SIAM J. Optim., 13 (2002), pp. 561–587.

[3] F. F. Clarke, *Optimization and Nonsmooth Analysis*, John Wiley & Sons, New York, 1983.

[4] A. L. Dontchev and W. W. Hager, *Implicit functions, Lipschitz maps, and stability in optimization*, Math. Oper. Res., 19 (1994), pp. 753–768.

[5] Z. Dostál and J. Schöberl, *Minimizing quadratic functions over non-negative cone with the rate of convergence and finite termination*, Comput. Optim. Appl., 30 (2005), pp. 23–44.

[6] C. Eck, J. Jarušek, and M. Krbec, *Unilateral Contact Problems. Variational Methods and Existence Theorems*, Pure Appl. Math. (Boca Raton) 270, Chapman & Hall/CRC, Boca Raton, FL, 2005.

[7] G. H. Golub and C. F. V. Loan, *Matrix Computation*, The Johns Hopkins University Press, Baltimore, MD, 1996.

[8] J. Haslinger, Z. Dostál, and R. Kučera, *Signorini problem with a given friction based on the reciprocal variational formulation*, in Nonsmooth/Nonconvex Mechanics: Modeling, Analysis and Numerical Methods, D. Y. Gao, R. W. Ogden, and G. E. Stavroulakis, eds., Kluwer Academic Publishers, Dordrecht, 2001, pp. 141–172.

[9] J. Haslinger, I. Hlaváček, and J. Nečas, *Numerical methods for unilateral problems in solid mechanics*, in Handbook of Numerical Analysis, Vol. IV, P. G. Ciarlet and J. L. Lions, eds., North–Holland, 1996, Amsterdam, pp. 313–485.

[10] J. Haslinger and P. Neittaanmäki, *Finite Element Approximation for Optimal Shape, Material and Topology Design*, 2nd ed., John Wiley & Sons, Chichester, 1996.

[11] J. Haslinger and P. D. Panagiotopoulos, *The reciprocal variational approach to the Signorini problem with friction. Approximation results*, Proc. Roy. Soc. Edinburgh Sect. A, 98 (1984), pp. 365–383.

[12] N. Kikuchi and J. T. Oden, *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods*, SIAM Stud. Appl. Math. 8, SIAM Press, Philadelphia, 1988.

[13] M. Kočvara and J. V. Outrata, *Optimization problems with equilibrium constraints and their numerical solution*, Math. Program. B, 101 (2004), pp. 119–150.

[14] R. Kučera, *Minimizing quadratic functions with separable quadratic constraints*, Optim. Methods Softw., 22 (2007), pp. 453–467.

[15] R. Kučera, *Convergence rate of an optimal algorithm for minimizing quadratic functions with separable convex constraints*, SIAM J. Optim., 19 (2008), pp. 846–862.

[16] Z.-Q. Luo, J.-S. Pang, and D. Ralph, *Mathematical Programs with Equilibrium Constraints*, Cambridge University Press, Cambridge, 1996.

[17] B. S. Mordukhovich, *Approximation Methods in Problems of Optimization and Control*, Nauka, Moscow, 1988 (in Russian).

[18] B. S. Mordukhovich, *Generalized differential calculus for nonsmooth and set-valued mappings*, J. Math. Anal. Appl. 183 (1994), pp. 250–288.

[19] B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation*, Vols. I and II, Springer-Verlag, Berlin-Heidelberg, 2006.

[20] J. Nečas and I. Hlaváček, *Mathematical Theory of Elastic and Elasto-Plastic Bodies: An Introduction*, Elsevier, Amsterdam, 1981.

[21] J. Outrata, M. Kočvara, and J. Zowe, *Nonsmooth Approach to Optimization Problems with Equilibrium Constraints: Theory, Applications and Numerical Results*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1998.

[22] J. V. Outrata, *A generalized mathematical program with equilibrium constraints*, SIAM J. Control Optim., 38 (2000), pp. 1623–1638.

[23] S. M. Robinson, *Strongly regular generalized equations*, Math. Oper. Res., 5 (1980), pp. 43–62.

[24] R. T. Rockafellar and R. Wets, *Variational Analysis*, Springer-Verlag, Berlin, 1998.

[25] H. Schramm and J. Zowe, *A version of the bundle idea for minimizing a nonsmooth function: Conceptual idea, convergence analysis, numerical results*, SIAM J. Optim., 2 (1992), pp. 121–152.

[26] J. Sokołowski and J.-P. Zolésio, *Introduction to Shape Optimization: Shape Sensitivity Analysis*, Springer-Verlag, Berlin, 1992.

[27] J. Zowe and H. Schramm, *Bundle Trust Methods: Fortran Codes for Nonsmooth Optimization. User's Guide*, Technical Report 259, Institute of Applied Mathematics, University of Erlangen, 2000.