Singular Perturbation Based Solution to Optimal Microalgal Growth Problem and Its Infinite Time Horizon Analysis

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Abstract—**The problem of the optimal microalgal growth of the so-called photosynthetic factory (PSF) is considered here. The objective is to maximize the photosynthetic production rate (the specific growth rate of microalgae) by manipulating the irradiance. Using the singular perturbation based reduction, an analytical solution of such an optimal control problem is obtained and its infinite horizon analysis shows that the optimal solution on large time intervals tends to the optimal steady state of PSF. This is a mathematical confirmation of the hypothesis often mentioned in biotechnological literature.**

Index Terms—**Photosynthetic factory (PSF).**

I. Introduction

The problem of the optimal control of bioreactors operating under the high irradiance belongs to intensively studied topics in both biotechnology and mathematical biology literature, see [7] and references within there. It is based on the photosynthetic microorganisms growth modelling reflecting the coupling between photosynthesis and irradiance (being a controlled input), resulting in the steady-state light response curve (so-called *P-I curve*), which represents the microbial kinetics, see e.g. *Monod* or *Haldane* type kinetics [14] and also survey introduction in [10].

Nevertheless, in order to study an optimal control of algae production, the dynamic model should be developed. The model considered later on is the lumped parameter model for photosynthesis and photoinhibition, the so-called model of photosynthetic factory—PSF model [2], [3], [5], [9], [17]. The main difficulty in considering the dynamic behavior of the photosynthetic processes consists in their different time scales. While the characteristic time of microalgal growth (e.g. doubling time) is in order of hours, light and dark reactions occur in milliseconds and photoinhibition in minutes, for more detail see e.g. [13].

The purpose of this technical note is to analyze the two time scales phenomena and to use this analysis to compute explicit optimal control law to maximize algal biomass production. Namely, the reduction of the dynamical system to the slow manifold will be developed and then the corresponding less dimensional optimal control problem will be solved analytically. As a matter of fact, this analytical solution being applied to the original non-reduced system even generates better values of the performance index than the approximation obtained via quite long numerical computations using the gradient algorithm, thereby confirming viability of the above reduction to the less dimensional problem. Moreover, further analysis of that analytical solution will give mathematical confirmation to the well-known biotechnological experimental observation and the paradigm that for large time intervals optimal solutions tend to be constant.

This technical note is organized as follows. Section II presents the dynamic model of the microalgal growth in detail, derives its reduction to the slow manifold and carefully analyzes the corresponding approximation precision. Section III applies Pontryagin's maximum principle to derive analytically the optimal irradiance to maximize the average production rate. It also formulates and proves some biotechnological relevant properties of the optimal solution. Conclusions are summarized in the final section.

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II. Dynamical Model of Microalgal Growth and Its Reduction

Microalgal growth is modelled based on the following experimental observations: (**i**) the steady state kinetics is of *Haldane* type [8]; (**ii**) the microalgal culture in suspension has the so-called *light integration* property [8], [15], i.e. as the light/dark cycle frequency, [4], is going to infinity, the value of the resulting production rate (e.g. oxygen evolution rate) goes to a certain limit value, which depends on the average irradiance only [9]. These features are best comprised by the dynamical model, called as the **model of photosynthetic factory (PSF)**, which has been recently studied in the biotechnological literature [2], [3], [5], [17].

PSF is a phenomenological model depicted schematically at Fig. 1. Here, every algae cell is assumed to be in the exactly one of the following three states: the activated one, the inhibited one and the resting one. These states are denoted A, B, R, respectively. Further, under the irradiance each algae cell with certain probabilities either stays in its current state or is transformed into one of the remaining states. It is assumed that transition rates depend on intensity of the irradiance in affine way. As a consequence, the PSF model can be mathematically described by the well-known concept from control theory—the so-called bilinear controlled dynamical system, cf. [1] and references within there. To do so, the state variables x_A, x_R, x_B are introduced as probabilities of the corresponding states. Obviously, as $x_R + x_A + x_B = 1$ by definition, it is sufficient to study any couple of these variables and transition between them. The states x_A and x_B, unlike x_R, can be directly measured and therefore they are usually preferred. This leads to the following controlled bilinear system with two dimensional state and one dimensional input:

$$\begin{bmatrix} \dot{x_A} \\ \dot{x_B} \end{bmatrix} = \begin{bmatrix} -\gamma & 0 \\ 0 & -\delta \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} + u(t) \begin{bmatrix} -(\alpha+\beta) & -\alpha \\ \beta & 0 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} + u(t) \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \quad (1)$$

where the constants $\alpha = 1.935 \times 10^{-3}\ \mu\mathrm{E}^{-1}\mathrm{m}^2$, $\gamma = 1.460 \times 10^{-1}\ \mathrm{s}^{-1}$, $\beta = 5.785 \times 10^{-7}\ \mu\mathrm{E}^{-1}\mathrm{m}^2$, $\delta = 4.796 \times 10^{-4}\ \mathrm{s}^{-1}$ are taken from [12], [17] and $u(t)$ is a known piecewise smooth scalar input representing the irradiance in $\mu\mathrm{Em}^{-2}\mathrm{s}^{-1}$, where μE stands for micro Einstein being 10^{-6} of the energy of 1 mol of fotons. This is common irradiance unit used for photosynthesis, as its rate depends on number of involved photons rather then on the precise energy. Energy of one μE depends on the wave length via the well known relation including Planck constant, for the 400 nm light 1 $\mu\mathrm{E} \approx 0.27$ J. Recall, that the state variables $x_{A,B}$ are **dimensionless**. Therefore, a straightforward physical dimension analysis shows that $\alpha, \beta, \gamma u, \delta u$ are all in s^{-1}, i.e. both sides of the model (1) are in s^{-1} and all the above introduced units and equations are physically consistent.

The desired biotechnological production is proportional to the so-called specific growth rate, [2], [17], being, in turn, proportional to the number of transitions from the activated to the resting state, i.e.

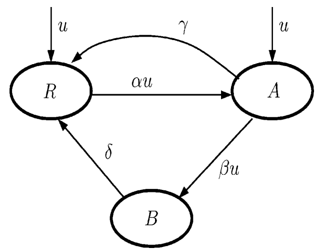

Fig. 1. Scheme of states and transition rates of the photosynthetic factory—Eilers and Peeters PSF model.

$\gamma x_A(t)$. More precisely, the specific growth rate measurable as the rate of the photosynthetic oxygen production is by [17] proportional to the integral average of the activated state. As a consequence, the performance index to be maximized is taken as the so-called average specific growth rate on a given fixed time interval $[t_0, t_f] \subset \mathbb{R}^+ \cup \{0\}$ defined as follows:

$$J = \kappa\gamma(t_f - t_0)^{-1} \int_{t_0}^{t_f} x_A(t) \mathrm{d}t. \quad (2)$$

Here κ is yet another dimensionless PSF model parameter which obviously does not influence the optimal solution. As a matter of fact, the average specific growth rate, measured in s^{-1}, characterizes the efficiency of the algae production process, nevertheless, its optimization on the fixed time interval is equivalent to optimizing the integral of the activated state.

For the constant input signal $u \geq 0$ the system of differential equation (1) is linear and its matrix has two distinct negative eigenvalues. Therefore, any solution of (1) with constant $u \geq 0$ globally converges to the following steady state solution depending on that constant $u \geq 0$:

$$x_{A_{ss}} = \alpha\delta u \lambda_F^{-1} \lambda_S^{-1}, \quad x_{B_{ss}} = \alpha\beta u^2 \lambda_F^{-1} \lambda_S^{-1} \quad (3)$$

where $\lambda_{F,S} < 0$ are eigenvalues of the corresponding constant matrix on the right hand side of (1). As already noted, the performance index to be maximized in the sequel is based on quantity defined in (2). If only constant irradiance is considered and steady state transition phenomena are neglected, an immediate idea is to maximize the steady state value $x_{A_{ss}}$ with respect to u. Straightforward computations [9], [12] show that such a maximal value exists and is achieved for the unique input denoted as $u_{opt_{ss}}$ and given as follows:

$$u_{opt_{ss}} = \gamma^{1/2}\delta^{1/2}\alpha^{-1/2}\beta^{-1/2}, \quad u^* := u/u_{opt_{ss}}. \quad (4)$$

In the sequel, with a slight abuse of notation, the above $u_{opt_{ss}}$ will be called as the **constant optimal input or control**. The variable u^* introduced in (4) is a new normalized input variable used in the sequel, with such an input variable the optimal constant input is simply equal to 1.

Next, let us prove that the above model (1) is not contradictory from a biological point of view. Namely, by their biological nature the state variables x_A, x_B are nonnegative and their sum should not exceed one as also $x_R = 1 - x_A - x_B$ is nonnegative. Therefore, it is vital to prove the following

Proposition 2.1: Denote

$$\Delta^1 := \left\{ [x_1, x_2]^\top \in \mathbb{R}^2 | x_1 + x_2 \leq 1, x_1 \geq 0, x_2 \geq 0 \right\}. \quad (5)$$

Then, Δ^1 is forward invariant with respect to all trajectories of (1) generated by any positive measurable input function $u(t)$.

Proof: Analyzing the right hand side of (1) gives easily that

1) For $x_A \in (0,1)$, $x_B = 0$, and by (1) $\dot{x}_B = u\beta x_A \geq 0$.
2) For $x_B \in (0,1)$, $x_A = 0$, and by (1) $\dot{x}_A = u\alpha(1 - x_B) \geq 0$.
3) For $x_A \in (0,1)$, $x_B \in (0,1)$, $x_A + x_B = 1$, and by (1) $\dot{x}_A + \dot{x}_B = -\gamma x_A - \delta x_B \leq 0$.
4) For $x_A = x_B = 0$ one has $\dot{x}_A = \alpha u$, $\dot{x}_B = 0$.
5) For $x_A = 0, x_B = 1$ one has $\dot{x}_A = 0$, $\dot{x}_B = -\delta$.
6) For $x_A = 1, x_B = 0$ one has $\dot{x}_A = -(\gamma + \beta u)$, $\dot{x}_B = \beta u$.

Therefore, the vector field of (1) does not point outward Δ^1 at any point of its boundary. □

Summarizing, the above described PSF model is a convenient modelling framework for the lumped parameter model of the microalgal growth satisfying two basic properties (i) and (ii) formulated at the beginning of the current section. The latter property (ii) is mathematically proved in [9] based on the earlier result on bilinear systems in [1]. For more details, see [9], [10], [12] and further references within there.

To facilitate further analysis let us rewrite the model (1), (2) introducing a more convenient parametrization. Namely, consider new parameters q_i, $i = 1, \ldots, 5$, defined as

$$q_1 := \sqrt{\frac{\gamma\delta}{\alpha\beta}}, \quad q_2 := \sqrt{\frac{\alpha\beta\gamma}{\delta}} \frac{1}{\alpha + \beta}, \; q_3 := \kappa\gamma\sqrt{\frac{\alpha\delta}{\beta\gamma}},$$
$$q_4 := \alpha q_1, \; q_5 := \beta/\alpha \quad (6)$$

together with the earlier introduced dimensionless irradiance $u^* := u/u_{opt_{ss}}$ giving the re-parameterized model

$$\frac{1}{q_4}\begin{bmatrix}\dot{x}_A \\ \dot{x}_B\end{bmatrix} = -\begin{bmatrix} q_2(1+q_5) & 0 \\ 0 & \frac{q_5}{q_2(1+q_5)}\end{bmatrix}\begin{bmatrix}x_A \\ x_B\end{bmatrix} - u^*\begin{bmatrix}(1+q_5) & 1 \\ -q_5 & 0\end{bmatrix}\begin{bmatrix}x_A \\ x_B\end{bmatrix} + \begin{bmatrix}u^* \\ 0\end{bmatrix}, \quad (7)$$

$$J = q_2 q_3 (1+q_5)(t_f - t_0)^{-1} \int_{t_0}^{t_f} x_A(t)\mathrm{d}t. \quad (8)$$

Notice that q_1 units are those of irradiance ($\mu\mathrm{E}\ \mathrm{m}^{-2}\ \mathrm{s}^{-1}$), q_2, q_5 are dimensionless, q_3, q_4 are in s^{-1}. The reason to introduce such a re-parameterization is that the role of each new parameter is now much more clearly visible. Namely, parameters q_1, q_2, q_3 correspond to the steady state properties of the PSF, while $q_1 := u_{opt_{ss}}$ by definition. Furthermore, q_4 influences the overall dynamics through a constant time scaling only, while q_5 is a small parameter quantifying the separation between the fast and slow dynamic; $q_5 \approx 10^{-4}$. More specifically, based on (1) and [17], the following values of the PSF re-parameterized model parameters were calculated and further tuned via special identification method in [12] for the microalga *Porphyridium* sp.: $q_1 := 250.106\ \mu\mathrm{Em}^{-2}\mathrm{s}^{-1}$, $q_2 := 0.301591$, $q_3 := 0.000176498\ \mathrm{s}^{-1}$, $q_4 := 0.483955\ \mathrm{s}^{-1}$, $q_5 := 0.000298966$. Finally, the expressions for the steady states depending on constant inputs given by (3) have after the above re-parameterization the following simpler form:

$$x_{B_{ss}} = u^{*2}\left(u^{*2} + u^*/q_2 + 1\right)^{-1},$$
$$x_{A_{ss}} = x_{B_{ss}}\left(q_2(1+q_5)u^*\right)^{-1}. \quad (9)$$

In particular, the constant input $u^* = 1$ maximizes the value of $x_{A_{ss}}$ among all constant $u^* \geq 0$.

Now, let us derive one dimensional reduction of the PSF model. The coefficients of the right hand side of the second row in (7) are by several orders smaller than those of the first one due to the presence of the small parameter $q_5 \approx 10^{-4}$. So, the idea is to replace the first much faster equation by its right hand side equal to zero, then to express x_A from it and to substitute the resulting expression into the second equation in (7). This gives the following reduced model (the upper index "S" aims to avoid confusion with notation for the non-reduced model (7)):

$${x_A}^S = \frac{u^*\left(1 - x_B^S\right)}{(u^* + q_2)(1+q_5)}, \quad (10)$$

$$\frac{\mathrm{d}x_B^S}{\mathrm{d}t} = -\frac{q_4 q_5 x_B^S}{q_2(1+q_5)} + \frac{q_4 q_5\left(1 - x_B^S\right)u^{*2}}{(1+q_5)(u^* + q_2)}. \quad (11)$$

The important question is how well this reduced model approximates the original non-reduced one. Apart from the system parameters, the approximation is obviously influenced by its time varying input, see [16] for general treatment of the related issues. The subsequent proposition and its corollary provide specific and reasonable estimates of the corresponding approximation precision based on the efficient exploiting the particular properties of (7), (10), and (11).

Proposition 2.2: Assume that $x(t, x^0)$, $t \in \mathcal{I} := [t_0, t_1]$, is the solution of (7) generated by the initial condition $x^0(t_0) = (x_A^0, x_B^0)^\top \in \Delta^1$ defined by (5) and a given positive bounded measurable function $u^*(t), t \in \mathcal{I}$. Further, let for $U_{ap} \in [0,1]$, $P > 0$, $D > 0$, $\varepsilon > 0$ and $\forall t \in \mathcal{I}$

$$\left|\frac{u^*(t)}{u^*(t)+q_2} - U_{ap}\right| \leq D, \quad \left|x_A{}^0 - U_{ap}\frac{1-x_B^0}{1+q_5}\right| \leq P,$$
$$t_1 - t_0 > T(\varepsilon) = (D+1)(q_2q_4)^{-1} \times \log\left(\varepsilon^{-1}\widetilde{K}(P-\overline{K})\right), \tag{12}$$
$$\widetilde{K} = \sqrt{2q_5^2 + 6q_5 + 5},$$
$$\overline{K} = \max\left\{D + q_5, q_5\frac{(D+1)^2}{4q_2^2}\right\}. \tag{13}$$

Suppose $q_{2,3,4,5} \geq 0, q_2 < 1$. Then there exists the solution $x^S(t, \tilde{x}^0)$, $x^S := (x_A^S, x_B^S)^\top$ of (10), (11) such that for all $\varepsilon > 0$ and $P > \overline{K}$ it holds

$$\left\|x^S(t,\tilde{x}^0) - x(t,x^0)\right\| < \widetilde{K}(\overline{K}+D)+\varepsilon, \quad \forall\, t \geq t_0 + T(\varepsilon). \tag{14}$$

Moreover, if $P \leq \overline{K}$ then it holds $\|x^S(t,\tilde{x}^0) - x(t,x^0)\| < \widetilde{K}(\overline{K}+D), \forall t \geq t_0$.

Proof: First, recall that by Proposition 2.1 the set Δ^1 is forward invariant, i.e. in the sequel we may assume that any trajectory belongs for all $t \in \mathcal{I}$ to Δ^1. Further, denote

$$e_1 = x_A - x_A^S,\ e_2 = x_B - x_B^S,$$
$$\epsilon_1 = x_A - \frac{(1-x_B)u^*}{(u^*+q_2)(1+q_5)}, \epsilon_1^{ap} = x_A - U_{ap}\frac{1-x_B}{1+q_5} \tag{15}$$

then by (10) and by $x_B \in [0,1]$ one has

$$e_1 = \epsilon_1 - \frac{e_2 u^*}{(u^*+q_2)(1+q_5)},$$
$$\epsilon_1 - \epsilon_1^{ap} = \frac{1-x_B}{1+q_5}\left[U_{ap} - \frac{u^*}{u^*+q_2}\right], |\epsilon_1 - \epsilon_1^{ap}| \leq D. \tag{16}$$

To estimate $\epsilon_1(t)$, $t \in \mathcal{I}$, notice that by the first row in (7) it holds $\dot{x}_A = -(u^*+q_2)q_4(1+q_5)\epsilon_1$, i.e. by (15), (16)

$$\dot{\epsilon}_1^{ap} = \frac{U_{ap}\dot{x}_B}{1+q_5} - \frac{(u^*+q_2)q_4}{(1+q_5)^{-1}} \times \left[\epsilon_1^{ap} + \frac{1-x_B}{1+q_5}\left[U_{ap} - \frac{u^*}{u^*+q_2}\right]\right],$$
$$\dot{\epsilon}_1^{ap} = -(u^*+q_2)q_4(1+q_5) \times \left[\epsilon_1^{ap} + \frac{1-x_B}{1+q_5}\left[U_{ap} - \frac{u^*}{u^*+q_2}\right]\right] - U_{ap}q_4q_5\frac{x_B - u^*q_2x_A(1+q_5)}{q_2(1+q_5)^2}$$

where the last equality is by the second row of (7). Summarizing, it holds

$$\dot{\epsilon}_1^{ap} = -(u^*+q_2)q_4(1+q_5)\left(\epsilon_1^{ap} + \kappa(t)\right), \tag{17}$$
$$\kappa(t) := q_5U_{ap}\frac{x_B - u^*q_2x_A(1+q_5)}{q_2(u^*+q_2)(1+q_5)^3} + \frac{1-x_B}{1+q_5}\left[U_{ap} - \frac{u^*}{u^*+q_2}\right]. \tag{18}$$

Moreover, the following estimate holds:

$$|\kappa(t)| \leq \overline{K} \quad \forall [x_A, x_B]^\top \in \Delta^1 \tag{19}$$

where Δ^1 is given by (5) and $\overline{K}$ by (13). Actually, $\kappa(t)$ depends linearly on $[x_A, x_B]^\top$ and therefore it is sufficient to check the estimate (19) on the vertices of Δ^1:

1) for $[x_A, x_B]^\top = [0,0]^\top$ it holds that $|\kappa(t)| \leq D$;
2) for $[x_A, x_B]^\top = [1,0]^\top$ it holds that $|\kappa(t)| \leq D + q_5U_{ap} \leq D + q_5$, as $U_{ap} \in [0,1]$;
3) for $[x_A, x_B]^\top = [0,1]^\top$ it holds that $|\kappa(t)| \leq (U_{ap}q_5/q_2(u^* + q_2)) \leq q_5(U_{ap}(1-U_{ap}+D)/q_2^2) \leq q_5[(D+1)/2q_2]^2$.

In 3) we used that $(u^*/(u^*+q_2)) \geq U_{ap} - D \Rightarrow u^* \geq (q_2(U_{ap} - D)/(1-U_{ap}+D)) \Rightarrow (1/(u^*+q_2)) \leq ((1-U_{ap}+D)/q_2)$ and $s(D+1-s) \leq (D+1)^2/4, \forall s \in \mathbb{R}$. To finish the estimating of $\epsilon_1^{ap}(t)$, (17) and (19) imply that the following inequalities hold simultaneously:

$$\dot{\epsilon}_1^{ap} \leq -(u^*+q_2)q_4(1+q_5)\left(\epsilon_1^{ap} - \overline{K}\right),$$
$$\dot{\epsilon}_1^{ap} \geq -(u^*+q_2)q_4(1+q_5)\left(\epsilon_1^{ap} + \overline{K}\right). \tag{20}$$

By integration and by Bellman-Gronwall lemma one has (note that $\overline{K} > 0$ by definition)

$$\epsilon_1^{ap}(t) \leq \overline{K} + \alpha(t)\left(\epsilon_1^{ap}(t_0) - \overline{K}\right) \leq \overline{K} + (P-\overline{K})\alpha(t),$$
$$\alpha(t) := \exp\left(-\int_{t_0}^{t}(q_2+u^*(s))\,q_4(1+q_5)\mathrm{d}s\right) \in (0,1],$$
$$\epsilon_1^{ap}(t) \geq -\overline{K} + \alpha(t)\left(\epsilon_1^{ap}(t_0) + \overline{K}\right) \geq -\overline{K} - (P-\overline{K})\alpha(t)$$
$$\Longrightarrow -\epsilon_1^{ap}(t) \leq \overline{K} + (P-\overline{K})\alpha(t).$$

By definition of ϵ_1^{ap} in (15) and definition of P in (12) it holds $\forall [x_A(t_0), x_B(t_0)]^\top \in \Delta^1$

$$|\epsilon_1^{ap}(t)| \leq \overline{K} + (P-\overline{K})\alpha(t). \tag{21}$$

As a consequence, $\forall \varepsilon > 0$ and $\forall [x_A(t_0), x_B(t_0)]^\top \in \Delta^1$ it obviously holds that $|\epsilon_1^{ap}(t)| < \overline{K} + \varepsilon\widetilde{K}^{-1}$ if

$$\int_{t_0}^{t}(q_2+u^*(s))\,q_4(1+q_5)\mathrm{d}s > \log\left(\varepsilon^{-1}\widetilde{K}(P-\overline{K})\right).$$

Using already derived estimate $(1/(u^*+q_2)) \leq ((1-U_{ap}+D)/q_2)$ together with the obvious inequality $(1/(1+q_5)) < 1$ one has easily that $|\epsilon_1^{ap}(t)| < \overline{K} + \varepsilon\widetilde{K}^{-1}$ if

$$t \geq t_0 + (1-U_{ap}+D)(q_2q_4)^{-1}\log\left(\varepsilon^{-1}\widetilde{K}(P-\overline{K})\right). \tag{22}$$

Recalling the relation between ϵ and ϵ^{ap} in (16) and $\widetilde{K}, T(\varepsilon)$ given by (12), (13), one has by (22)

$$|\epsilon_1(t)| < \overline{K} + D + \varepsilon/\widetilde{K}, \quad \forall t \geq t_0 + T(\varepsilon). \tag{23}$$

So far we have shown that any trajectory of (7) gets after sufficiently large time arbitrarily close to $\overline{K}$-proximity of the slow manifold. To finish the proof, one should show that there is a solution of (10), (11), which is sufficiently closed to that trajectory of (7) for all times $t \geq T(\varepsilon)$. To do so, subtract (10), (11) and (7) to obtain after some straightforward computations and by (23) that $\forall t \geq t_0 + T(\varepsilon)$

$$\dot{e}_2 = u^*q_4q_5\left[\epsilon_1 - e_2\frac{q_2+u^*+q_2(u^*)^2}{q_2(1+q_5)(u^*+q_2)u^*}\right],$$
$$|\epsilon_1| < \overline{K} + D + \frac{\varepsilon}{\widetilde{K}}.$$

Now, one can define the solution $x^S(t,\tilde{x}^0)$ of (10), (11) involved in (14): this is simply the solution of (10), (11) satisfying $e_2(t_0+T(\varepsilon)) = 0$, where e_2 is given by (15). Notice, that both the original non-reduced system and its restriction to the slow manifold obviously satisfy for any bounded measurable input the conditions for the existence and uniqueness of solutions for all $t \in \mathbb{R}$. As a consequence, the condition $e_2(t_0 + T(\varepsilon)) = 0$ defines uniquely and globally in time the solution of (10), (11), thereby actually defining $\tilde{x}^0$ required by the proposition claim. With such a selection of initial condition for x^S (and consequently for e_2) it is not difficult to see that

$$|e_2(t)| \leq \left(\overline{K} + D + \frac{\varepsilon}{\widetilde{K}}\right) \max_{u^* \in [U_l, U_h]} \left[\frac{q_2(1+q_5)(u^* + q_2)u^*}{q_2 + u^* + q_2(u^*)^2}\right]$$
$$\leq (1+q_5)\left(\overline{K} + D + \frac{\varepsilon}{\widetilde{K}}\right), \quad \forall t \geq t_0 + T(\varepsilon)$$

as $q_2 \in (0,1)$ by assumption, i.e. $q_2(u^*)^2 + q_2^2 u^* \leq q_2 + u^* + q_2(u^*)^2$. To conclude the proof, note that, in addition to the estimate just obtained, by (23), (15) it holds $\forall t \geq t_0 + T(\varepsilon)$

$$|e_1(t)| = \left|\epsilon_1(t) - \frac{e_2(t)u^*}{(u^* + q_2)(1+q_5)}\right|$$
$$\leq (2 + q_5)\left(\overline{K} + D + \frac{\varepsilon}{\widetilde{K}}\right).$$

Finally, notice, that if $P \leq \overline{K}$ then following (21) one can in all estimates replace ε by 0 and $T(\varepsilon)$ by 0. Now, computations using $\|e\| = \sqrt{e_1^2 + e_2^2}$ complete the proof. □

Corollary 2.3: Suppose that all assumptions and notation of Proposition 2.2 hold except that the constant U_{ap} is replaced by a piecewise continuous function $U_{ap}(t) \in [0,1] \forall t \in \mathcal{I}$ such that its jumps at discontinuities have absolute value less than $E > 0$ and time segments between jumps are longer than $\Delta T := (D+1)(q_2 q_4)^{-1}\log(2)$. Then for $P > \overline{K}$ it holds $\forall t \geq t_0 + T(\varepsilon)$

$$\left\|x^S(t, \tilde{x}^0) - x(t, x^0)\right\| < \widetilde{K}(\overline{K} + D + 2E) + \varepsilon. \qquad (24)$$

Moreover, if $P \leq \overline{K}$ then it holds $\|x^S(t, \tilde{x}^0) - x(t, x^0)\| < \widetilde{K}(\overline{K} + D + 2E), \forall t \geq t_0$.

Proof: Let $t^1 < \ldots < t^k$ be the jumps time moments, where $t^1, t^k \in [t_0, t]$. The estimate (21) is in this case replaced by (here $\beta(s) := (q_2 + u^*(s))q_4(1+q_5)$ and recall that $t^{i+1} - t^i \geq \Delta T$)

$$|\epsilon_1^{ap}(t)| \leq \overline{K} + (P - \overline{K})\mathrm{e}^{-\int_{t_0}^{t} \beta(s)\mathrm{d}s}$$
$$+ \mathrm{e}^{-\int_{t^k}^{t} \beta(s)\mathrm{d}s} E \sum_{i=0}^{k-1} \mathrm{e}^{-\int_{t^{k-i}}^{t^k} \beta(s)\mathrm{d}s}$$
$$\leq \overline{K} + (P - \overline{K})\mathrm{e}^{-\int_{t_0}^{t} \beta(s)\mathrm{d}s}$$
$$+ E\mathrm{e}^{-\int_{t^k}^{t} \beta(s)\mathrm{d}s} \sum_{i=0}^{k-1} 2^{-i}$$
$$\leq \overline{K} + (P - \overline{K})\mathrm{e}^{-\int_{t_0}^{t} \beta(s)\mathrm{d}s} + 2E$$

since $\forall i = 0, 1, \ldots, k-1$ it holds $\int_{t^{k-i}}^{t^k} \beta(s)\mathrm{d}s \geq i\Delta T(D+1)^{-1} q_2 q_4 = i\log(2)$. Roughly saying, each jump causes additional error in $|\epsilon_1^{ap}|$ less than E which decays before the next jump at least by one half, so in total, even infinite many jumps cause an additional increase of $|\epsilon_1^{ap}(t)|$ of no more than $2E = (1 + 1/2 + 1/4 \ldots)E$. Now, one can proceed with the latter estimate in the same way as in the rest of the proof of Proposition 2.2 following after (21). □

Remark 2.4: Corollary 2.3 shows that the singular perturbation approximates quite well also the systems with discontinuous or even only measurable inputs, provided they are approximated by piecewise constant functions with reasonable jumps and reasonable times between them. The minimal time between jumps ΔT can be estimated for given values of q_2, q_4 as $\Delta T \approx (D+1)4.8$ s, while $\widetilde{K} \approx 5, \overline{K} \approx \max\{D + 0.003, 0.0003\}$. Important observation is the following: the approximation is better if function $u^*(t)/(u^*(t) + q_2)$ is better approximated in the above sense, not the input $u^*(t)$ itself. Fig. 2 shows that $u^*(t)/(u^*(t) + q_2)$ becomes quickly saturated for $u^* \approx 1$, i.e. the approximation condition of the above Corollary

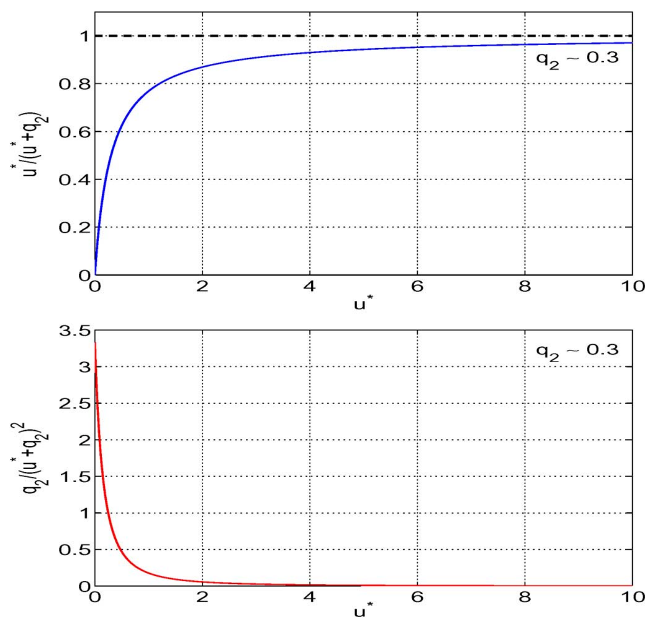

Fig. 2. Saturated influence of the input on the approximation precision. Left: the function $\boldsymbol{u/(u+q_2)}$, right: its derivative.

2.3 is easier to fulfill, especially recalling the fact that $u^* \equiv 1$ is constant optimal control and the nonconstant one is by experiments expected to be $\geq$1. Finally, the minimal time between jumps is indeed very small, as typical time segments where optimality is investigated are of order 10^5 s (i.e., several days). Moreover, there is an obvious relation between ΔT and D, E: e.g., when approximating a fixed input with bounded variation, smaller ΔT means also smaller D, E. Summarizing, the singular perturbation based reduction is acceptable for reasonable slowly changing inputs and might be used to find some nonconstant optimal control strategies later on.

III. Optimal Control—Maximum Principle for Reduced System

In this section, the optimal control problem for the system (11) with the performance index obtained by substituting x_A^S from (10) for x_A in (8), is considered and solved analytically. Recall that the initial state is assumed to be given and fixed, the final state is free and time interval is fixed. To make the further exposition more standard, consider without any loss of generality $t_0 = 0, t_1 = T$, the minimization of the integral in (8) with the minus sign added and denote $x_1 := x_B$, $u := u^*$. Summarizing, for the given **fixed** $T > 0, U > 0, x^0 \in \mathbb{R}^2$, the following optimal control problem is to be solved: find measurable on $[0, T]$ function $u(t)$ such that:

$$J = \int_0^T (x_1 - 1)\frac{u(t)}{u(t) + L}\mathrm{d}t \mapsto \min, \; u(t) \in [0, U], \qquad (25)$$

$$\dot{x_1} = -\frac{K}{L}x_1 + \frac{(1-x_1)u^2}{u+L}K, \; x_1(0) = x_1^0 \in [0,1],$$
$$K := q_4 q_5 (1+q_5)^{-1}, \; L := q_2. \qquad (26)$$

First, notice that in (25), (26) it always holds $x_1(t) \in [0,1], \forall t \geq 0$. To solve this problem, the Pontryagin Maximum Principle (PMP) [11], formulated for the interested reader's convenience in a more particular and simpler form as Proposition 1 of [10], will be applied. The Hamiltonian $\mathcal{H}$ and the adjoint system for (25), (26) are

$$\mathcal{H} = -\frac{u(x_1 - 1)}{u + L} + \psi_1 K\left(\frac{(1-x_1)u^2}{u+L} - \frac{1}{L}x_1\right), \qquad (27)$$

$$\dot{\psi}_1 = \frac{u}{u+L} + \psi_1\left(\frac{K}{L} + u\frac{u}{u+L}K\right), \psi_1(T) = 0. \qquad (28)$$

Suppose $u^o(t)$, $t \in [0,T]$, solves the optimal control problem (25), (26), then by PMP it holds for all $t \in [0,T]$ that $\mathcal{H}(\psi_1(t), u^o(t), x(t)) = \max_{u\in[0,U]} \mathcal{H}(\psi_1(t), u, x(t)) \equiv 0$, for some solution of (28) $\psi_1(t) \not\equiv 0$, i.e. $\phi(u^0) = \max_{u\in[0,U]} \phi(u)$, $\phi(u) := u(1-x_1)(1+\psi_1 K u)/(u+L)$, where $\psi_1(t)$ is the uniquely given solution of (28). To determine u^0, compute $\phi'(u)$ to obtain

$$\frac{\partial\phi(u)}{\partial u} = \frac{1-x_1}{(u+L)^2}(K\psi_1 u^2 + 2KL\psi_1 u + L). \quad (29)$$

First of all, it is obvious from (25), (26) that $x_1(t) < 1 \forall t > 0$. As the co-state ψ_1 is given by (28), it is easy to see that $\psi_1(t) \leq 0 \forall t < T$. Actually, assuming $\psi_1(t') > 0$ for some $t' < T$ one has by (recall that $K, L > 0$) $(u/(u+L)) \geq 0$, $(K/L) + u(u/(u+L))K > 0, \forall u \in [0,U]$, that $\dot\psi_1(t) > 0, \forall t \geq t'$, i.e. $\psi_1(t) > 0, \forall t \geq t'$ what contradicts to the condition $\psi_1(T) = 0$. From the same equation one can see that $\psi_1 \equiv 0$ on some $[t', T]$ if and only if $u(t) \equiv 0, \forall t \in [t', T]$. Nevertheless, on such a time interval the derivative in (29) equals to $L > 0$, i.e. maximum of ϕ can not be achieved at $u = 0$. Therefore, only the case $\psi_1(t) < 0 \forall t < T$ is possible. Now, using (29) for $\psi_1 < 0$ and $x_1 \in [0,1]$ (cf. remark right after (26)) one can see that $\phi'(u) > 0$, $u \in [0, \tilde u]$, $\phi'(\tilde u) = 0$, $\phi'(u) < 0$, $u \in [\tilde u, \infty]$, where $\tilde u(\psi_1) = -L + \sqrt{L^2 - (K\psi_1)^{-1}}$.

Summarizing, the only possible optimal control $u^o(t)$ is given by the following formula:

$$u^o(t) = \alpha(\psi_1(t)), \alpha(\psi_1) = \min\left\{-L + \sqrt{L^2 - \frac{L}{K\psi_1}}, U\right\} \quad (30)$$

where $\psi_1(t)$ is the solution of (28) with $u = \alpha(\psi_1)$.

As a matter of fact, to obtain the optimal control (30) one has first to solve a nonlinear differential equation, i.e. (28) with $u = \alpha(\psi_1)$ and then substitute this solution to the above $\alpha(\cdot)$. Such a nonlinear equation obviously does not have the solution in the closed form and may be solved only numerically. Nevertheless, all crucial qualitative properties of this solution can be obtained by rigorous theoretical analysis of that nonlinear equation (28) with $u = \alpha(\psi_1)$. First, the full qualitative description of the above optimal control is formulated and proved as the following

Proposition 3.1: Optimal control given in (30) is strictly increasing on time interval $[0, T - T^{sat}]$ while on $[T - T^{sat}, T]$ it holds $u(t) \equiv U$. Moreover, the length T^{sat} of the interval where the optimal control is saturated does not depend on T, namely, it equals to

$$T^{sat} = \frac{L(U+L)}{K(U+L+LU^2)} \log\left(\frac{U^2(U+2L)}{(U^2-1)(U+L)}\right).$$

Proof: The first part of the proposition follows from the fact that the right hand side of the adjoint equation is always strictly positive, so that $\psi_1(t) < 0 \forall t < T$ and strictly increases, while in (30) u^o depends on ψ_1 in strictly increasing way, unless the saturation occurs. To obtain the formula for T^{sat}, consider in (30) the costate ψ_1^{sat} where $-L + \sqrt{L^2 - (K\psi_1^{sat})^{-1}} = U$, i.e.

$$\psi_1^{sat} := \frac{-L}{KU(U+2L)}. \quad (31)$$

As $\psi_1(T) = 0$, $\psi_1(T^{sat}) = \psi_1^{sat}$ and $\psi_1(t)$ is strictly increasing, T^{sat} should obviously satisfy the following relation $0 = \mathrm{e}^{K((1/L)+(U^2/(U+L)))T^{sat}}(\psi_1^{sat} + \int_0^{T^{sat}}(U\mathrm{e}^{-K((1/L)+(U^2/(U+L)))s}/(U+L))\mathrm{d}s)$, i.e. after integration, re-grouping and cancelling some terms one has that $(\mathrm{e}^{((K/L)+U(U/(U+L))K)T^{sat}} - (U^2(U+2L)/(U^2-1)(U+L))) = 0$, giving easily the above formula for T^{sat}. □

Corollary 3.2: The optimal control does not depend on initial state of the plant.

Proof: It is obvious from the corresponding formulas defining the optimal control. Notice, that the optimal value of the performance index **depends** on initial condition, nevertheless, it is achieved with the same input, regardless of the initial condition. □

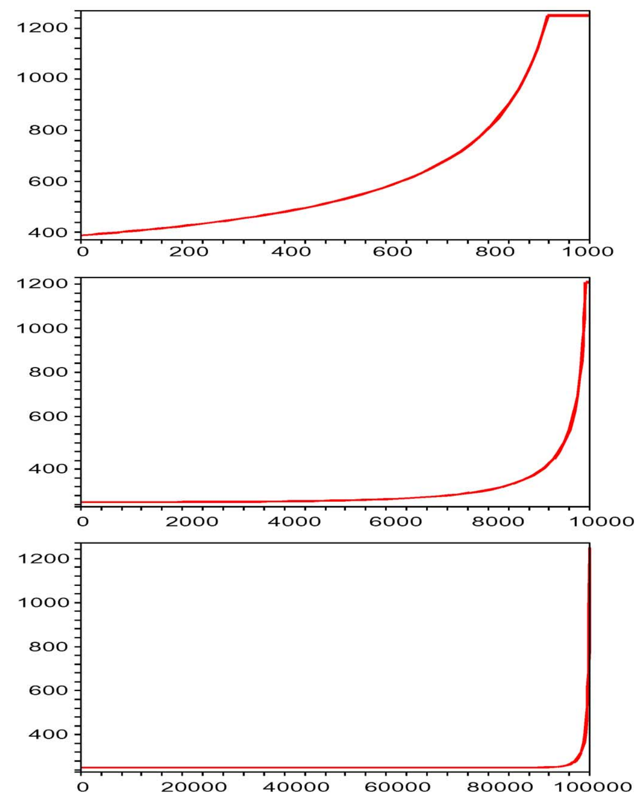

Fig. 3. SP based optimal control in $\boldsymbol{\mu}\mathbf{E}\ \mathbf{m}^{-2}\mathbf{s}^{-1}$; time in s.

The proposition just proved and its corollary show that the optimal control course depends only on the input saturation and does not depend on initial condition $x_1(0)$. Besides, for the same U and two different $T_1 > T_2$ the optimal control on $[0, T_1]$ coincides on subinterval $[T_1 - T_2, T_1]$ with the optimal control on interval $[0, T_2]$. Moreover, for $U \geq 1$ with increasing T, the optimal control converges to the constant input $u \equiv 1$ known to maximize the performance index within constant inputs. More precisely, it holds the following

Proposition 3.3: Denote $u_T^o(t)$ the optimal control (30) corresponding to the fixed time interval $[0,T]$ and assume $U \geq 1$. Then $\forall \epsilon, \tilde T > 0, \exists T(\epsilon, \tilde T) > 0 : |u^o_{T(\epsilon,\tilde T)}(t) - 1| \leq \epsilon, \forall t \in [0, \tilde T]$.

Proof: Consider the following relations:

$$\begin{aligned} \dot\psi_1 &= \frac{\alpha(\psi_1)}{\alpha(\psi_1)+L} + \psi_1\left(\frac{K}{L} + \frac{K\alpha^2(\psi_1)}{\alpha(\psi_1)+L}\right), \\ &\quad \psi_1(T) = 0, \end{aligned} \quad (32)$$

$$\begin{aligned} \alpha(\psi_1) &= \min\left\{-L + \sqrt{L^2 - L(K\psi_1)^{-1}}, U\right\}, \\ \psi_1^\epsilon &:= -L\,(K(1+2L))^{-1}, \quad \alpha(\psi_1^\epsilon) = 1. \end{aligned} \quad (33)$$

Straightforward, though laborious computations show that $\alpha(-L/K(1+2L)) = \min\{\sqrt{L^2+2L+1} - L, U\} = \min\{1, U\} = 1$, $(1/(1+L)) + (-L/K(1+2L))((K/L) + (K/(1+L))) = 0$. Therefore, ψ_1^ϵ given by (33) is the equilibrium of (32) being, in turn, the co-state (29) with $u = \alpha(\psi_1)$. Further

$$\frac{\alpha(\psi_1)}{\alpha(\psi_1)+L} + \psi_1\left(\frac{K}{L} + \frac{K\alpha^2(\psi_1)}{\alpha(\psi_1)+L}\right)\begin{cases} < 0 & \text{for } \psi_1^\epsilon > \psi \\ = 0 & \text{for } \psi_1^\epsilon = \psi \\ > 0 & \text{for } \psi_1^\epsilon < \psi \end{cases}$$

giving by the simple Lyapunov-like function $V = (\psi - \psi^{eq})^2/2$ argument that the equilibrium (33) is actually the unique and globally asymptotically stable one for the system (32) in reversed time. The last fact obviously implies that $\forall \tilde T > 0, \forall \epsilon > 0 \exists T = T(\tilde T, \epsilon) : |\psi_1(t) - \psi_1^\epsilon| < \epsilon, t \in [0, \tilde T]$, where $\psi_1(t)$ is the solution of (32). Now,

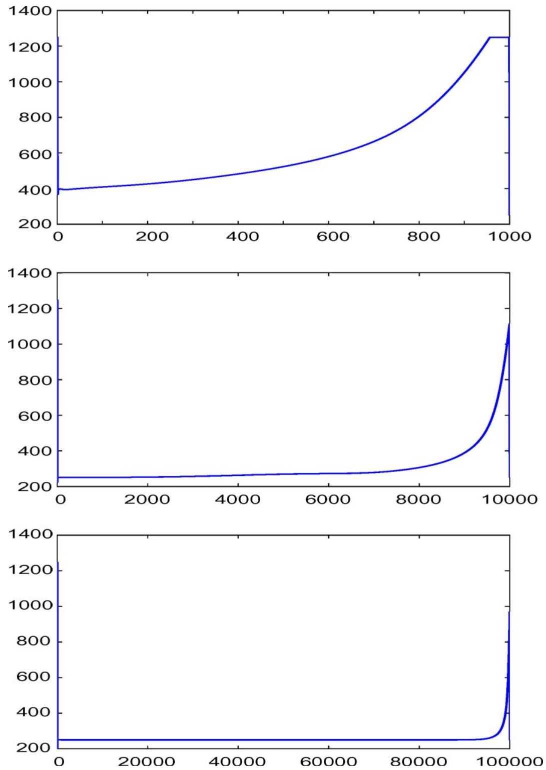

Fig. 4. Optimal control computed numerically in $\mu\mathbf{E}\ \mathbf{m}^{-2}\mathbf{s}^{-1}$; time in s.

TABLE I
VALUES OF $(1/T)\int_0^T x_A(t)\mathrm{d}t$ FOR (1) WITH DIFFERENT INPUTS

T in s	$u \equiv 250\mu$E	Fig. 3	Fig. 4
10^3	0.442	0.479	0.4796
10^4	0.5830	0.5893	0.58927
10^5	0.61951	0.62020	0.62013

the claim of the proposition to be proved follows by the second equality of (33) and by (30). □

Remark 3.4: Proposition 3.3 shows rigorously that on large time intervals the constant optimal control is closed to the general nonconstant optimal control, moreover the latter one does not depend on the initial condition. This is actually the well-known and experimentally confirmed conjecture, yet until now without rigorous justification. Figs. 3, 4 and Table I nicely illustrate these properties. The analytical solution based on the singular perturbation reduction is compared with the numerical results based on the gradient numerical algorithm. Moreover, Table I compares values of the performance index, showing that on a short interval the "constant optimal control" is significantly worse than both reduced and gradient based ones. Finally, notice, that on longer time intervals the reduced-based optimal control is actually even better than the one provided by gradient numerical algorithm. Here, for the sake of a fair comparison, the reduced-based optimal control is applied to the **non-reduced** system, so it is indeed better. The reason is the well known phenomenon: the slow convergence of the gradient algorithm when approximating the point of saturation, see also Figs. 3, 4, where the notable difference can be seen around the saturation points.

IV. CONCLUSION

In this technical note, the problem of the optimal production rate for algae photosynthetic factory has been analyzed. The main conclusion here is that on sufficiently large time interval the optimal control is closed to the constant one, which optimizes the appropriate component of the system steady state. This fact has been mathematically established both analytically based on singular perturbation reduction and numerically via gradient optimization algorithm. In such a way, this technical note provides the mathematical confirmation of the experimentally based hypothesis that has been frequently mentioned in biotechnological literature.

REFERENCES

[1] S. Čelikovský, "On the continuous dependence of trajectories of bilinear systems on controls and its applications," *Kybernetika*, vol. 24, pp. 278–292, 1988.

[2] P. H. C. Eilers and J. C. H. Peeters, "A model for the relationship between light intensity and the rate of photosynthesis in phytoplankton," *Ecol. Model.*, vol. 42, pp. 199–215, 1988.

[3] P. H. C. Eilers and J. C. H. Peeters, "Dynamic behaviour of a model for photosynthesis and photoinhibition," *Ecol. Model.*, vol. 69, pp. 113–133, 1993.

[4] M. Janssen, P. Slenders, J. Tramper, L. R. Mur, and R. H. Wijffels, "Photosynthetic efficiency of Dunaliella tertiolecta under short light/dark cycles," *Enzyme Microb. Technol.*, vol. 29, pp. 298–305, 2001.

[5] T. Kmet', M. Straškraba, and P. Mauersberger, "A mechanistic model of the adaptation of phytoplankton photosynthesis," *Bull. Math. Biol.*, vol. 55, pp. 259–275, 1993.

[6] H. K. Khalil, "Stability analysis of singularly perturbed systems," in *Singular Perturbations and Asymptotic Analysis in Control Systems*. Berlin, Germany: Springer, 1987.

[7] J. Masojídek, Š. Papáček, V. Jirka, J. Červený, J. Kunc, J. Korečko, M. Sergejevová, O. Verbovikova, J. Kopecký, D. Štys, and G. Torzillo, "A closed solar photobioreactor for cultivation of microalgae under supra-high irradiances: Basic design and performance of pilot plant," *J. Appl. Phycol.*, vol. 15, pp. 239–248, 2003.

[8] L. Nedbal, V. Tichý, F. Xiong, and J. U. Grobbelaar, "Microscopic green algae and cyanobacteria in high-frequency intermittent light," *J. Appl. Phycol.*, vol. 8, pp. 325–333, 1996.

[9] Š. Papáček, S. Čelikovský, D. Štys, and J. Ruiz-León, "Bilinear system as modelling framework for analysis of microalgal growth," *Kybernetika* vol. 43, pp. 1–20, 2007.

[10] Š. Papáček, S. Čelikovský, and J. Ruiz-León, "Optimal feedback control of microalgal growth based on the slow reduction," in *Proc. IFAC World Congress*, Seoul, Korea, Jul. 2008, pp. 14588–14593.

[11] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mischenko, *The Mathematical Theory of Optimal Processes*. New York: Wiley, 1962.

[12] B. Rehák, S. Čelikovský, and Š. Papáček, "Model for photosynthesis and photoinhibition: Parameter identification based on the harmonic irradiation O_2 response measurement," *Joint Spec. Issue Syst. Biol., IEEE Trans. Circuits Syst. I & IEEE Trans. Autom. Control*, vol. 54, no. 1, pp. 101–108, Jan. 2008.

[13] R. Rosello Sastre, Z. Csögör, I. Perner-Nochta, P. Fleck-Schneider, and C. Posten, "Scale-down of microalgae cultivations in tubular photo-bioreactors-A conceptual approach," *J. Biotechnol.*, vol. 132, pp. 127–133, 2007.

[14] K. Schugerl and K.-H. Bellgardt, *Bioreaction Engineering, Modeling and Control*. Berlin, Germany: Springer-Verlag, 2000.

[15] K. L. Terry, "Photosynthesis in modulated light: Quantitative dependence of photosynthetic enhancement on flashing rate," *Biotechnol. Bioeng.*, vol. 28, pp. 988–995, 1986.

[16] A. Vigodner, "Limits of singularly perturbed control problems with statistical dynamics of fast motions," *SIAM J. Control Optim*, vol. 35, pp. 1–27, 1997.

[17] X. Wu and J. C. Merchuk, "A model integrating fluid dynamics in photosynthesis and photoinhibition processes," *Chem. Eng. Sci.*, vol. 56, pp. 3527–3538, 2001.