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ON THE AUBIN PROPERTY OF CRITICAL POINTS TO PERTURBED SECOND-ORDER CONE PROGRAMS*

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Abstract. We characterize the Aubin property of a canonically perturbed KKT system related to the second-order cone programming problem in terms of a strong second-order optimality condition. This condition requires the positive definiteness of a quadratic form, involving the Hessian of the Lagrangian and an extra term, associated with the curvature of the constraint set, over the linear space generated by the cone of critical directions. Since this condition is equivalent with the Robinson strong regularity, the mentioned KKT system behaves (with some restrictions) similarly as in nonlinear programming.

Key words. second-order cone programming, strong regularity, Aubin property, strong second-order sufficient optimality conditions, nondegeneracy

AMS subject classifications. 90C, 90C31, 90C46

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1. Introduction.

In nonlinear programming we have (for sufficiently smooth data) an equivalence between the strong second-order sufficient optimality condition (SSOSC) and linear independence constraint qualification condition (LICQ) on one hand, and Robinson's strong regularity of the canonically perturbed Karush–Kuhn–Tucker (KKT) system at the solution on the other hand (cf. [24, Theorem 4.1] and [8, Theorem 4.10]). Thanks to the results in Dontchev and Rockafellar [9, Theorems 1, 4, and 5], one further knows that these properties are equivalent with the Aubin property of the perturbed KKT system at the solution. This last equivalence is somewhat surprising because, at first glance, the Aubin property seems to be much less restrictive than the strong regularity. In fact, this equivalence is violated whenever the problem data are only in $C^{1,1}$ (continuously differentiable with Lipschitz gradient; cf. [16]) or provided we have to do with noncanonical perturbations (cf. [14]).

This paper is devoted to the study of the following nonlinear *second-order cone programming problem (SOCP)*:

$$\text{(SOCP)} \qquad \underset{x\in\mathbb{R}^n, s^j\in\mathbb{R}^{m_j+1}}{\mathrm{Min}} f(x); \quad g^j(x) = s^j, \quad (s^j)_0 \geq \|\bar{s}^j\|, \quad j = 1, \dots, J,$$

where f and g^j, $j = 1, \dots, J$, are twice continuously differentiable mappings from $\mathbb{R}^n$ into $\mathbb{R}$ and $\mathbb{R}^{m_j+1}$, respectively. Here we use the standard convention of indexing components of vectors of $\mathbb{R}^{m_j+1}$ from 0 to m_j, and given $s \in \mathbb{R}^{m_j+1}$, $\bar{s}$ denotes the subvector $(s_1, \dots, s_{m_j})^\top$. The vectors in $\mathbb{R}^n$ are indexed in the standard way from 1 to n, and by $\|\cdot\|$ we denote the Euclidean norm. The *second-order cone* (or *ice-cream cone*, or *Lorentz cone*) of dimension $m + 1$ is defined to be

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$$Q_{m+1} \coloneqq \{s \in \mathbb{R}^{m+1};\ s_0 \geq \|\bar{s}\|\}.$$

Hence, SOCP can be written as

$$\operatorname*{Min}_x f(x); \quad g^j(x) \in Q_{m_j+1}, \quad j = 1, 2, \dots, J,$$

or

$$\operatorname*{Min}_x f(x); \quad g(x) \in \mathcal{Q},$$

where $g(\cdot) \coloneqq (g^1(\cdot), \dots, g^J(\cdot))$ and $\mathcal{Q} \coloneqq \prod_{j=1}^J Q_{m_j+1}$. Clearly, under a constraint qualification (CQ), one can associate with SOCP a KKT system in the form of a variational inequality (VI), or a generalized equation (GE), with a nonpolyhedral constraint set. The solution map of this VI, when perturbed in a canonical way, assigns the perturbing parameter the KKT (critical) pairs of SOCP.

In Bonnans and Ramírez [5], the authors have proved a counterpart of the first equivalence above in which the SSOSC has been appropriately changed to consider the curvature of the constraint set, and LICQ has been replaced by the *nondegeneracy* condition, standard in conical programming. In the present paper we intend to study a possible extension of the second equivalence to the SOCP framework.

As in [9] our main workhorse is the characterization of the Aubin property via the so-called Mordukhovich criterion; see formula (2.9) below. We, however, do not establish a counterpart of [9, Theorem 1], but reduce in the first step the perturbed KKT system to a single GE

$$\eta \in Df(x) + (Dg(x))^\top N_Q(g(x)) \tag{1.1}$$

in variable x only. Then we prove that the Aubin property of the solution map $\eta \mapsto x$ in (1.1) around $(0, x^*)$ is equivalent with SSOSC *under* the nondegeneracy condition and a restriction on the position of the KKT pair. This gives raise to a certain *reduced* form of the second equivalence in nonlinear programming, which is the main result of this paper. Our proof makes use of the following two essential ingredients:

(i) An explicit formula for the limiting (Mordukhovich) coderivative of the metric projection onto Q_{m+1}, which has been computed in Outrata and Sun [22], and

(ii) a generalization of the second-order chain rule from Mordukhovich and Outrata [21, Theorem 3.4].

Besides the mentioned paper [9] a useful characterization of the Aubin property for inverses of strongly B-differentiable PC^1 functions has been provided in [13, Theorem 3.5] in terms of the so-called coherent orientation condition. This result, however, does not completely cover the case of SOCP, due to the nature of Q_{m+1}.

There are many important optimization problems arising in structural design (e.g., [2]), support vector machines and data classification (e.g., [2], [26]), robust optimization (e.g., [3], [17]), and in numerous other areas, which can be advantageously modeled or reformulated as SOCP; see [1] (and the references therein) for an overview of such examples, as well as for a review of the theoretical properties of SOCP problems. So, the investigation of the behavior of the critical points of SOCP with respect to the considered perturbations is important both in postoptimal analysis as well as in a possible control of solutions via parameters.

The paper is organized as follows. Section 2 is devoted to preliminaries. To facilitate the reading, it is split into subsections 2.1–2.4. In section 2.1 the reader finds a few basic facts about the Jordan algebra associated with Q_{m+1}. This algebra is the main tool for studying SOCP. In section 2.2 we provide the definitions of several notions from modern variational analysis used in what follows. Furthermore, we present the main result from [22], i.e., the mentioned formula for the coderivative of the projection onto Q_{m+1}. This subsection is closed by a new second-order chain rule that enables us to compute the coderivative of normal-cone mappings associated with the constraints system of SOCP. Section 2.3 deals with first-order optimality conditions. Here we define the mappings S and S^e, whose local behavior is the main object of this study. Finally, in section 2.4 we recall the notion of strong regularity and its connection with SSOSC, taken over from [5] and [28]. Our main results are stated in section 3. Therein, Theorem 21 concerns SOCP with $J = 1$. For this case, under nondegeneracy, the Aubin property is characterized in terms of SSOSC. In this way we establish a similar relationship between the Aubin property and the strong regularity as in [9]. Theorem 26 concerns SOCP in its general form. For this case, the desired characterization is proven provided that, for the local minimizer x^* and its corresponding (unique) Lagrange multilplier y^*, there exists at most one block j satisfying either $g^j(x^*) = 0$ and $y^{*j} \in \partial Q_{m_j+1} \setminus \{0\}$ or $g(x^*) \in \partial Q \setminus \{0\}$ and $y^{*j} = 0$ or $g^j(x^*) = 0 = y^{*j}$. Finally, section 5 contains some short conclusions.

The following notation is employed. For a set A, symbols ∂A, $\mathrm{Sp}(A)$, and $\mathrm{lin}(A)$ stand for its boundary, span, and the lineality space, respectively. The latter, $\mathrm{lin}(A)$, is defined as the biggest linear space contained in A. For a single-valued mapping f, $Df(x)$ and $D^2 f(x)$ denote its first- and second-order derivatives at x. $\mathbb{B}$ is the unit ball and I is the identity matrix. Sometimes we write I_n to indicate the dimension. $o(\cdot)$ denotes, as usual, a function from $\mathbb{R}_+$ to $\mathbb{R}$ with the property that $\lim_{t\to 0} t^{-1} o(t) = 0$. Finally, $T_A(s) := \{d : s + td + o(t) \in A \text{ for all } t > 0\}$ stands for the *tangent cone* to a set A at the point $s \in A$.

2. Preliminaries.

2.1. Algebra preliminaries on SOCP.

In this section, we recall some basic concepts and properties about the Jordan algebra associated with the second-order cone Q_{m+1} that are needed for this work (see [11] for more details). For any $v = (v_0, \bar{v})$, $w = (w_0, \bar{w}) \in \mathbb{R} \times \mathbb{R}^m$, the *Jordan product* for Q_{m+1} is defined by

$$v \circ w = (\langle v, w\rangle, v_0 \bar{w} + w_0 \bar{v}), \tag{2.1}$$

where $\langle v, w\rangle = v^\top w = \sum_{j=0}^m v_j w_j$. This product can be equivalently written as

$$v \circ w = \mathrm{Arw}(v) w, \tag{2.2}$$

where

$$\mathrm{Arw}(v) := \begin{pmatrix} v_0 & \bar{v}^\top \\ \bar{v} & v_0 I_m \end{pmatrix}$$

is the *arrow* matrix of vector v. In the case of block vectors v, w in $\Pi_{j=1}^J \mathbb{R}^{m_j+1}$ we set

$$v \circ w := \mathrm{vec}(v^1 \circ w^1, \ldots, v^J \circ w^J),$$

where, for any block vector $u = (u^1, \ldots, u^J)$ in $\Pi_{j=1}^J \mathbb{R}^{m_j+1}$, $\mathrm{vec}(u)$ denotes the row vector $(u^1, \ldots, u^J)^\top$.

It is clear that the product $\circ$ is commutative, and it follows easily from (2.2) that $(v, w) \mapsto v \circ w$ is a bilinear function where the element $e = (1, 0, \ldots, 0) \in \mathbb{R}^{m+1}$ plays the role of unit element for this algebra. On the other hand, the cone Q_{m+1} is not closed under the product $\circ$.

We recall the well-known property (e.g., [1, Lemma 15]):

$$\text{For all } v, w \in Q_{m+1}, \quad v \circ w = 0 \text{ if and only if } \langle v, w \rangle = 0. \tag{2.3}$$

Moreover, it is easily checked that relations in (2.3) are satisfied if and only if v and w belong to Q_{m+1} and

$$\begin{aligned} &\text{either } v = 0 \quad \text{or} \quad w = 0, \quad \text{or there exists } \alpha > 0 \text{ such that} \\ &v_0 = \alpha w_0 \quad \text{and} \quad \bar{v} = -\alpha \bar{w}. \end{aligned} \tag{2.4}$$

We next introduce the *spectral decomposition* of vectors in $\mathbb{R}^{m+1}$ associated with Q_{m+1}. For any $w = (w_0, \bar{w}) \in \mathbb{R} \times \mathbb{R}^m$, we can decompose w as

$$w = \lambda_1(w) c_1(w) + \lambda_2(w) c_2(w), \tag{2.5}$$

where $\lambda_1(w)$, $\lambda_2(w)$ and $c_1(w)$, $c_2(w)$ are the *spectral values* and *spectral vectors* of w given by

$$\lambda_i(w) = w_0 + (-1)^i \|\bar{w}\|, \tag{2.6}$$

$$c_i(w) = \begin{cases} \frac{1}{2}\left(1, (-1)^i \frac{\bar{w}}{\|\bar{w}\|}\right) & \text{if } \bar{w} \neq 0, \\ \frac{1}{2}(1, (-1)^i \bar{v}) & \text{if } \bar{w} = 0 \end{cases} \tag{2.7}$$

for $i = 1, 2$ with $\bar{v}$ being any unit vector in $\mathbb{R}^m$. If $\bar{w} \neq 0$, the decomposition (2.5) is unique. Notice that $\lambda_1(w) \leq \lambda_2(w)$ and the vectors $c_i(w)$, $i = 1, 2$ belong to ∂Q_{m+1}.

For each $w = (w_1, \bar{w}) \in \mathbb{R} \times \mathbb{R}^m$, the *trace* and the *determinant* of w with respect to Q_{m+1} are defined as

$$\begin{aligned} \operatorname{tr}(w) &= \lambda_1(w) + \lambda_2(w) = 2w_0, \\ \det(w) &= \lambda_1(w)\lambda_2(w) = w_0^2 - \|\bar{w}\|^2. \end{aligned} \tag{2.8}$$

These definitions can be viewed as the analogues of the trace and the determinant of matrices.

The spectral decomposition entails some basic properties, which are summarized below.

Proposition 1. *For any $w = (w_0, \bar{w}) \in \mathbb{R} \times \mathbb{R}^m$ the spectral values $\lambda_1(w)$, $\lambda_2(w)$ and spectral vectors $c_1(w)$, $c_2(w)$ given by* (2.6) *and* (2.7) *have the following properties:*

(a) *$c_1(w)$ and $c_2(w)$ are orthogonal for the Jordan product; i.e., $c_1(w) \circ c_2(w) = 0$, and $\|c_1(w)\| = \|c_2(w)\| = \frac{1}{\sqrt{2}}$.*

(b) *$c_1(w)$ and $c_2(w)$ are idempotent for the Jordan product: $c_i(w) \circ c_i(w) = c_i(w)$ for $i = 1, 2$.*

(c) *$\lambda_1(w)$, $\lambda_2(w)$ are nonnegative (resp., positive) if and only if $w \in Q_{m+1}$ (resp., $w \in \operatorname{int} Q_{m+1}$).*

(d) *The Euclidean norm of w can be represented as $\|w\|^2 = \frac{1}{2}(\lambda_1(w)^2 + \lambda_2(w)^2)$.*

For a proof of the previous proposition, see [1], [11].

2.2. Nonsmooth analysis. Consider a set $A \subseteq \mathbb{R}^n$ and a multifunction $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$. Recall that the *graph* of Φ is defined by $\operatorname{Gr}\Phi := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m | y \in \Phi(x)\}$.

Definition 2. *Let $\bar{x} \in A$ and $(\bar{x}, \bar{y}) \in \operatorname{Gr}\Phi$.*

(i) *The regular (Fréchet) normal-cone to A at $\bar{x}$, denoted by $\hat{N}_A(\bar{x})$, is defined by*

$$\hat{N}_A(\bar{x}) := \{x^* \in \mathbb{R}^n | \langle x^*, x - \bar{x}\rangle \leq o(\|x - \bar{x}\|) \,\forall\, x \in A\}.$$

(ii) *The limiting (Mordukhovich) normal-cone to A at $\bar{x}$, denoted by $N_A(\bar{x})$, is the cone*

$$\begin{aligned} N_A(\bar{x}) &:= \operatorname*{Limsup}_{x \to \bar{x}} \hat{N}_A(x) \\ &= \{x^* \in \mathbb{R}^n | \exists x_k \xrightarrow{A} \bar{x}, x_k^* \to x^* \textit{ such that } x_k^* \in \hat{N}_A(x_k)\}. \end{aligned}$$

(iii) *The (limiting) coderivative of Φ at $(\bar{x}, \bar{y})$ is the multifunction $D^*\Phi(\bar{x}, \bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ defined for all $y^* \in \mathbb{R}^m$ by*

$$D^*\Phi(\bar{x}, \bar{y})(y^*) := \{x^* \in \mathbb{R}^n | (x^*, -y^*) \in N_{\operatorname{Gr}\Phi}(\bar{x}, \bar{y})\}.$$

Remark 3. When the set A is convex, both normal-cones defined in (i) and (ii) coincide with the classical normal-cone used in convex analysis: $N_A(\bar{x}) := \{x^* : \langle x^*, x\rangle \leq \langle x^*, \bar{x}\rangle \text{ for all } x \in A\}$.

The limiting notions above ((ii) and (iii)) admit a rich calculus and play an important role in modern variational analysis. For their properties and respective calculus rules the reader is referred to [19] and the monographs [25], [20]. The limiting coderivative enables us to characterize an important Lipschitz-like behavior of multifunctions, called the Aubin property.

Definition 4. *Multifunction Φ has the Aubin property around $(\bar{x}, \bar{y})$, provided there are neighborhoods $\mathcal{U}$ of $\bar{x}$ and $\mathcal{V}$ of $\bar{y}$ and a positive scalar L such that*

$$\Phi(x_1) \cap \mathcal{V} \subseteq \Phi(x_2) + L\|x_1 - x_2\|\mathbb{B} \qquad \forall x_1, x_2 \in \mathcal{U}.$$

It turns out that Φ possesses the Aubin property around $(\bar{x}, \bar{y})$ if and only if

$$D^*\Phi(\bar{x}, \bar{y})(0) = \{0\}; \tag{2.9}$$

cf. [18, Proposition 3.5]. For different proofs and an extensive discussion on this subject, see also [25, Theorem 9.40], [20, Theorem 4.10]. In [25], the condition (2.9) is called the *Mordukhovich criterion.*

Let us now analyze the metric projection onto Q_{m+1}, which will be denoted by P. It is easy to see that P is differentiable (even continuously differentiable) at points $z \in \mathbb{R}^{m+1}$, where $\det(z) \neq 0$. One has that (cf. [15])

$$DP(z) = \begin{cases} 0 & \text{if } z_0 < -\|\bar{z}\|, \\ I_{m+1} & \text{if } z_0 > +\|\bar{z}\|, \\ \frac{1}{2}\begin{pmatrix} 1 & \bar{w}^\top \\ \bar{w} & H \end{pmatrix} & \text{if } -\|\bar{z}\| < z_0 < +\|\bar{z}\|, \end{cases}$$

where $\bar{w} := \frac{\bar{z}}{\|\bar{z}\|}$ and $H := (1 + \frac{z_0}{\|\bar{z}\|})I_m - \frac{z_0}{\|\bar{z}\|}\bar{w}\bar{w}^\top$.

Due to the symmetry of DP at these points

$$D^*P(z, P(z))(z^*) = DP(z)z^* \quad \forall\, z^* \in \mathbb{R}^{m+1}.$$

The main result of [22] consists in the computation of D^*P at points $(z, P(z))$, where $\det(z) = 0$. To present this result in a simple form, consider first a fixed z with $\bar{z} \neq 0$ and define the matrices

$$A(z) := I + \frac{1}{2}\begin{bmatrix} -1 & \frac{\bar{z}^\top}{\|\bar{z}\|} \\ \frac{\bar{z}}{\|\bar{z}\|} & -\frac{\bar{z}(\bar{z})^\top}{\|\bar{z}\|^2} \end{bmatrix} = \mathrm{Proj}_{(c_1(z))^\perp}(\cdot) \tag{2.10}$$

and

$$B(z) := \frac{1}{2}\begin{bmatrix} 1 & \frac{\bar{z}^\top}{\|\bar{z}\|} \\ \frac{\bar{z}}{\|\bar{z}\|} & \frac{\bar{z}(\bar{z})^\top}{\|\bar{z}\|^2} \end{bmatrix} = I - \mathrm{Proj}_{(c_2(z))^\perp}(\cdot). \tag{2.11}$$

THEOREM 5. *Let $z \in \mathbb{R}^{m+1}$ have the spectral decomposition as in* (2.5) *and let $u^* \in \mathbb{R}^{m+1}$. Then one has*

(i) *if* $\det(z) = 0$ *but* $\lambda_2(z) \neq 0$, *i.e.,* $z \in \partial Q_{m+1} \setminus \{0\}$, *then*

$$D^*P(z)(u^*) = \begin{cases} \mathrm{conv}\{u^*, A(z)u^*\} & \text{if } \langle u^*, c_1(z)\rangle \geq 0, \\ \{u^*, A(z)u^*\} & \text{otherwise}; \end{cases} \tag{2.12}$$

(ii) *if* $\det(z) = 0$ *but* $\lambda_1(z) \neq 0$, *i.e.,* $z \in \partial(-Q_{m+1}) \setminus \{0\}$, *then*

$$D^*P(z)(u^*) = \begin{cases} \mathrm{conv}\{0, B(z)u^*\} & \text{if } \langle u^*, c_2(z)\rangle \geq 0, \\ \{0, B(z)u^*\} & \text{otherwise}. \end{cases} \tag{2.13}$$

So, it remains to consider the most difficult case $z = 0$. In order to keep the respective formula for $D^*P(0)$ in a concise form, we employ also the notion of the *B-Jacobian* of P, defined by

$$\bar{\partial}_B P(z) := \left\{\lim_{k\to+\infty} DP(z_k) \,|\, z_k \to z,\ P \text{ is differentiable at } z_k\right\}.$$

THEOREM 6. *For all $u^* \in \mathbb{R}^{m+1}$ one has*

$$\begin{aligned} D^*P(0)(u^*) = &\bar{\partial}_B P(0)u^* \cup (Q_{m+1} \cap u^* - Q_{m+1}) \\ &\cup \bigcup_{A\in\mathcal{A}} \mathrm{conv}\{u^*, Au^*\} \cup \bigcup_{B\in\mathcal{B}} \mathrm{conv}\{0, Bu^*\}, \end{aligned} \tag{2.14}$$

where

$$\bar{\partial}_B P(0) = \{I, 0\} \cup \left\{\frac{1}{2}\begin{bmatrix} 1 & w^\top \\ w & 2\alpha I + (1-2\alpha)ww^\top \end{bmatrix} \,\middle|\, w \in \mathbb{R}^{n-1},\ \|w\| = 1,\ \alpha \in [0,1]\right\}, \tag{2.15}$$

and

$$\mathcal{A} := \left\{ I + \frac{1}{2}\begin{bmatrix} -1 & w^\top \\ w & -ww^\top \end{bmatrix} | w \in \mathbb{R}^{n-1},\ \|w\| = 1,\ \left\langle u^*, \begin{bmatrix} 1 \\ -w \end{bmatrix} \right\rangle \geq 0 \right\},$$
$$\mathcal{B} := \left\{ \frac{1}{2}\begin{bmatrix} 1 & w^\top \\ w & ww^\top \end{bmatrix} | w \in \mathbb{R}^{n-1}, \|w\| = 1, \left\langle u^*, \begin{bmatrix} 1 \\ w \end{bmatrix} \right\rangle \geq 0 \right\}.$$

In the last part of this subsection we will be dealing with the constraint set

$$A := \{x \in \mathbb{R}^n | g(x) \in K\}, \tag{2.16}$$

where the map $g: \mathbb{R}^n \to \mathbb{R}^m$ is twice continuously differentiable and $K \subset \mathbb{R}^m$ is a general closed and convex set. In section 3 we then put K equal to Q_{m+1} or to the Cartesian product of second-order cones, but Theorem 7 below can also be used in other situations and therefore we will present it in a general form.

Let $\bar{x} \in A$ be a reference point and put $\bar{s} := g(\bar{x})$. Assume the existence of a twice continuously differentiable *reduction* map $h: \mathbb{R}^m \to \mathbb{R}^k$ and a closed convex set $\mathcal{C} \subset \mathbb{R}^k$ such that

$$\text{(R)} \qquad \left.\begin{array}{l} 1.\ h(\bar{s}) = 0,\ Dh(\bar{s}) \text{ is surjective and there is a neighborhood } \mathcal{O} \\ \text{of } \bar{s} \text{ such that} \\ \qquad\qquad K \cap \mathcal{O} = h^{-1}(\mathcal{C}) \cap \mathcal{O}; \\ 2.\ T_{\mathcal{C}}(h(\bar{s})) \text{ is pointed.} \end{array}\right\}$$

In agreement with [7, Definition 3.135], condition (R) means that K is (C^2-) *reducible* to the set $\mathcal{C}$ at $\bar{s}$, and the reduction is pointed. Further, following [7, Definition 4.40], we will suppose that $\bar{x}$ is a *nondegenerate* point of the mapping g with respect to K and h; i.e.,

$$Dg(\bar{x})\mathbb{R}^n + \operatorname{Ker} Dh(\bar{s}) = \mathbb{R}^m. \tag{2.17}$$

As shown in [7, section 4.6], under the pointedness assumption above, (2.17) is equivalent with the condition

$$Dg(\bar{x})\mathbb{R}^n + \operatorname{lin}(T_K(\bar{s})) = \mathbb{R}^m \tag{2.18}$$

or, equivalently, with the implication

$$\left.\begin{array}{l} (Dg(\bar{x}))^\top \lambda = 0 \\ \lambda \in \operatorname{Sp} N_K(\bar{s}) \end{array}\right\} \Rightarrow \lambda = 0, \tag{2.19}$$

which is independent of the reduction mapping h. So, under (R), either of the conditions (2.18), (2.19) can be used as definition of nondegeneracy. This definition is discussed in a general conic programming framework in [7, section 4.6] and extends, from classical nonlinear programming, the concept of linear independence of gradients of active constraints. Note that other known definitions of nondegeneracy in the SOCP literature are essentially equivalent; see, for instance, [1, Definition 18] and the references therein.

On the basis of (2.19) it can easily be proved that to each $\bar{v} \in N_A(\bar{x})$, there exists a unique multiplier $\bar{y} \in N_K(\bar{s})$ such that $\bar{v} = (Dg(\bar{x}))^\top \bar{y}$. Moreover, $D(h \circ g)(\bar{x})$ is surjective.

Our aim is now to compute the coderivative of normal-cone mapping $N_A(\cdot)$, which plays a crucial role in the main results of this paper. In the proof of the respective statement below we make use of the following formula for differentiation of a special composition.

Let C be an $[l \times p]$ matrix whose entries are continuously differentiable functions of $x \in \mathbb{R}^n$. Similarly, let the elements of $b \in \mathbb{R}^p$ be continuously differentiable functions of x. Then one has

$$D(C(\cdot)b(\cdot))|_{x=\bar{x}} = D(C(\cdot)b(\bar{x}))|_{x=\bar{x}} + D(C(\bar{x})b(\cdot))|_{x=\bar{x}}. \tag{2.20}$$

THEOREM 7. *Let $\bar{x}$ be a nondegenerate point of g with respect to K and h, and assume that the conditions* (R) *are fulfilled. Let $(\bar{x}, \bar{v}) \in \mathrm{Gr} N_A$ and $\bar{y} \in \mathbb{R}^m$ be the (unique) vector from $N_K(\bar{s})$ such that $\bar{v} = (Dg(\bar{x}))^\top \bar{y}$. Then for all $v^* \in \mathbb{R}^m$*

$$D^* N_A(\bar{x}, \bar{v})(v^*) = \left(\sum_{i=1}^{m} \bar{y}_i \, Dg_i^2(\bar{x})\right) v^* + Dg(\bar{x})^\top D^* N_K(\bar{s}, \bar{y})(Dg(\bar{x})v^*). \tag{2.21}$$

Proof. Denote $f = h \circ g$. By [21, Theorem 3.4] and our assumptions above, there exists a unique multiplier vector $\bar{\mu} \in N_{\mathcal{C}}(f(\bar{x}))$ with $(Df(\bar{x}))^\top \bar{\mu} = \bar{v}$ such that

$$D^* N_A(\bar{x}, \bar{v})(v^*) = \left(\sum_{i=1}^{k} \bar{\mu}_i \, D^2 f_i(\bar{x})\right) v^* + Df(\bar{x})^\top D^* N_{\mathcal{C}}(f(\bar{x}), \bar{\mu})(Df(\bar{x})v^*). \tag{2.22}$$

Clearly,

$$Df(\bar{x}) = D(h \circ g)(\bar{x}) = Dh(\bar{s})Dg(\bar{x})$$

and

$$Df(\bar{x})^\top = Dg(\bar{x})^\top Dh(\bar{s})^\top. \tag{2.23}$$

Moreover, since $Dh(\bar{s})$ is surjective, one has again by [21, Theorem 3.4] for all $\lambda^* \in \mathbb{R}^k$ the equality

$$D^* N_K(\bar{s}, \bar{y})(\lambda^*) = \left(\sum_{i=1}^{k} \bar{\vartheta}_i \, D^2 h_i(\bar{s})\right) \lambda^* + Dh(\bar{s})^\top D^* N_{\mathcal{C}}(h(\bar{s}), \bar{\vartheta})(Dh(\bar{s})\lambda^*),$$

where $\bar{\vartheta}$ is the unique vector from $\mathbb{R}^k$ satisfying the relations

$$\bar{\vartheta} \in N_{\mathcal{C}}(h(\bar{s})), \quad Dh(\bar{s})^\top \bar{\vartheta} = \bar{y}. \tag{2.24}$$

As $h(\bar{s}) = f(\bar{x})$, it follows by virtue of the uniqueness of $\bar{\mu}$ that $\bar{\vartheta} = \bar{\mu}$. Indeed, both $\bar{\vartheta}$ and $\bar{\mu}$ belong to $N_{\mathcal{C}}(f(\bar{x}))$, and by virtue of (2.23),

$$Df(\bar{x})^\top \bar{\vartheta} = Dg(\bar{x})^\top Dh(\bar{s})^\top \bar{\vartheta} = Dg(\bar{x})^\top \bar{y} = \bar{v}.$$

Taking this into account we observe that

$$Df(\bar{x})^\top D^* N_{\mathcal{C}}(f(\bar{x}),\bar{\mu})(Df(\bar{x})v^*) = Dg(\bar{x})^\top Dh(\bar{s})^\top D^* N_{\mathcal{C}}(h(\bar{s}),\bar{\vartheta})(Dh(\bar{s})Dg(\bar{x})v^*)$$
$$= Dg(\bar{x})^\top \left[D^* N_K(\bar{s},\bar{y})(Dg(\bar{x})v^*) - \left(\sum_{i=1}^k \bar{\mu}_i D^2 h_i(\bar{s})\right)(Dg(\bar{x})v^*)\right].$$

It remains thus to show that

$$\left(\sum_{i=1}^k \bar{\mu}_i D^2 f_i(\bar{x})\right) v^* = \left(\sum_{i=1}^m \bar{y}_i D^2 g_i(\bar{x})\right) v^* + Dg(\bar{x})^\top \left(\sum_{i=1}^k \bar{\mu}_i D^2 h_i(\bar{s})\right)(Dg(\bar{x})v^*).$$

To this end we invoke formula (2.20) and conclude that, taking $Df_i(\cdot)$ and $Dh_i(\cdot)$ as column vectors, one has

$$\begin{aligned} D(Df_i(\bar{x})) &= D(Dg(\cdot)^\top Dh_i(g(\cdot)))|_{x=\bar{x}} \\ &= D(Dg(\cdot)^\top Dh_i(\bar{s}))|_{x=\bar{x}} + Dg(\bar{x})^\top D(Dh_i(g(\cdot)))|_{x=\bar{x}}. \end{aligned}$$

Hence,

$$\begin{aligned} \left(\sum_{i=1}^k \bar{\mu}_i D^2 f_i(\bar{x})\right) v^* = &\left[D\left(Dg(\cdot)^\top \sum_{i=1}^k \bar{\mu}_i Dh_i(\bar{s})\right)\Bigg|_{x=\bar{x}}\right] v^* \\ &+ Dg(\bar{x})^\top \sum_{i=1}^k \bar{\mu}_i D^2 h_i(\bar{s})(Dg(\bar{x})v^*)\Bigg). \end{aligned} \tag{2.25}$$

Since $\sum_{i=1}^k \bar{\mu}_i Dh_i(\bar{s}) = \bar{y}$, the first term on the right-hand side of (2.25) amounts to

$$(D([Dg_1(\cdot), \ldots, Dg_m(\cdot)]\bar{y})|_{x=\bar{x}})v^* = \left(\sum_{i=1}^m \bar{y}_i D^2 g_i(\bar{x})\right) v^*$$

and we are done. □

Several remarks are in order.

Remark 8. Formula (2.21) has been derived in [21] under the surjectivity of $Dg(\bar{x})$. In such a case one does not need to take care about the reducibility of K. In this sense, (2.21) can be viewed as generalization of the respective chain rule from [21].

Remark 9. In [7, section 3.4] one finds important examples of reducible sets. The reducibility of Q_{m+1} or the Cartesian product of the second-order cones has been proved in [5, Lemma 15].

2.3. First-order optimality conditions for nonlinear SOCP. Let x^* be a (local) solution of SOCP satisfying Robinson's CQ condition which attains in this case the following form:

$$\text{There exists } h^* \in \mathbb{R}^n \text{ such that } g(x^*) + Dg(x^*)h^* \in \operatorname{int} \mathcal{Q}. \tag{2.26}$$

Under (2.26) there is a vector $y^* = (y^{*1}, \ldots, y^{*J})^\top$ of *Lagrange multipliers* associated with x^* such that the pair (x^*, y^*) fulfills the KKT conditions

$$\begin{aligned} &0 = D_x L(x,y), \\ &y^j \in Q_{m_j+1}, \quad \langle y^j, g^j(x)\rangle = 0 \quad \forall j = 1, \ldots, J, \end{aligned} \tag{2.27}$$

where

$$L(x, y) := f(x) - \sum_{j=1}^{J} \langle y^j, g^j(x) \rangle$$

is the *Lagrangian* function and $y = (y^1, \ldots, y^J)^\top$. We denote by $\Lambda(x^*)$ the set of vectors y that satisfy condition (2.27) with $x = x^*$. They are termed *Lagrange multipliers* associated with x^*. Under (2.26), $\Lambda(x^*)$ is a nonempty and compact set.

On the other hand, since the sets Q_{m_j+1}, $j = 1, 2, \ldots, J$ are self-dual cones, one has

$$\left.\begin{array}{c} g^j(x) \in Q_{m_j+1} \\ y^j \in Q_{m_j+1} \\ \langle y^j, g^j(x) \rangle = 0 \end{array}\right\} \Leftrightarrow -y^j \in N_{Q_{m_j+1}}(g^j(x)) \Leftrightarrow -g^j(x) \in N_{Q_{m_j+1}}(y^j). \tag{2.28}$$

So, Robinson's condition (2.26) can be equivalently stated as follows:

$$\left.\begin{array}{l} \sum_{j=1}^{J} (Dg^j(x^*))^\top y^j = 0 \\ y^j \in Q_{m_j+1} \\ \langle y^j, g^j(x^*) \rangle = 0, j = 1, \ldots, J \end{array}\right\} \Rightarrow y = 0. \tag{2.29}$$

Indeed, (2.26) is equivalent to saying that (see [7, Proposition 2.97 and Corollary 2.98])

$$Dg(x^*)\mathbb{R}^n + T_{\mathcal{Q}}(g(x^*)) = \mathbb{R}^{m+1} \tag{2.30}$$

or

$$\operatorname{Ker} Dg(x^*) \cap N_{\mathcal{Q}}(g(x^*)) = \{0\}. \tag{2.31}$$

Due to (2.28), condition (2.31) clearly coincides with condition (2.29). They amount to the *(generalized) Mangasarian–Fromowitz CQ* for SOCP; cf. [12].

It also follows from (2.28) that KKT conditions (2.27) can be written as the "enhanced" GE

$$\begin{aligned} 0 &= D_x L(x, y), \\ 0 &\in g(x) + N_Q(y). \end{aligned} \tag{2.32}$$

By virtue of (2.28), the variable y in (2.32) can be eliminated, and in this way we arrive at the GE

$$0 \in Df(x) + (Dg(x))^\top N_Q(g(x)) \tag{2.33}$$

in variable x only.

Remark 10. The GE (2.33) can also be constructed directly on the basis of the optimality condition

$$0 \in Df(x) + \hat{N}_\Gamma(x),$$

where Γ is the (generally nonconvex) set $g^{-1}(Q)$. It suffices to realize that under the posed assumptions

$$\hat{N}_\Gamma(x) = (Dg(x))^\top N_Q(g(x))$$

for x close to x^*; cf. [25, Exercise 10.26 (d)].

The solutions to (2.33), i.e., the x-parts of the solutions to (2.32), are called the *critical points* of SOCP. Clearly, under (2.26) each (local) solution of SOCP is a critical point.

Via canonical perturbations of GE (2.32) and (2.33), we define now the multifunctions

$$S^e\colon \mathbb{R}^n \times \prod_{j=1}^J \mathbb{R}^{m_j+1} \rightrightarrows \mathbb{R}^n \times \prod_{j=1}^J \mathbb{R}^{m_j+1} \quad \text{and} \quad S\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$$

by

$$\begin{aligned} S^e(\delta_1, \delta_2) &:= \{(x,y) | \delta_1 = D_x L(x,y), \delta_2 \in g(x) + N_Q(y)\}, \\ S(\eta) &:= \{x | \eta \in Df(x) + (Dg(x))^\top N_Q(g(x))\}. \end{aligned}$$

Multifunction S will be called the *critical point map* of SOCP. Both S^e and S play a crucial role in the further development. Clearly, $x \in S(\eta)$ if and only if there is a y such that

$$(x,y) \in S^e(\eta, 0). \tag{2.34}$$

The remainder of this section is devoted to two additional notions that are extensively used in the following. First, we recall from [5, Lemma 2.5] the characterization of the tangent cone to the second-order cone.

LEMMA 11. *Consider the second-order cone $Q := Q_{m+1}$ and let $s \in Q$. Then,*

$$T_Q(s) = \begin{cases} \mathbb{R}^{m+1} & \text{if } s \in \operatorname{int} Q, \\ Q & \text{if } s = 0, \\ d \in \mathbb{R}^{m+1} \colon \langle \bar{d}, \bar{s} \rangle - s_0 d_0 \le 0 & \text{if } s \in \partial Q \setminus \{0\}. \end{cases} \tag{2.35}$$

Another important set associated with a point x^*, feasible for SOCP, is termed the cone of critical directions at x^* and defined by

$$C(x^*) := Df(x^*)^\perp \cap Dg(x^*)^{-1} T_{\mathcal{Q}}(g(x^*)). \tag{2.36}$$

If $\Lambda(x^*)$ is nonempty, and $y^* \in \Lambda(x^*)$, then

$$C(x^*) = \{h \colon Dg(x^*)h \in T_{\mathcal{Q}}(g(x^*)) \cap (y^*)^\perp\}.$$

From now on, we use the notations $s^j := g^j(x^*)$ and $d^j(h) := Dg^j(x^*)h$ (for a given $h \in \mathbb{R}^n$).

COROLLARY 12. *Let x^* be a critical point of problem SOCP and $y^* \in \Lambda(x^*)$. Then, the cone of critical directions $C(x^*)$ is given by*

$$C(x^*) = \left\{ h \in \mathbb{R}^n \left| \begin{array}{ll} d^j(h) \in T_{Q_{m_j+1}}(s^j) & \textit{if } y^{*j} = 0 \\ d^j(h) = 0 & \textit{if } y^{*j} \in \operatorname{int} Q_{m_j+1} \\ d^j(h) \in \mathbb{R}_+(y_0^{*j}, -\bar{y}^{*j}) & \textit{if } y^{*j} \in \partial Q_{m_j+1} \setminus \{0\},\ s^j = 0 \\ \langle d^j(h), y^{*j} \rangle = 0 & \textit{if } \ y^{*j},\ s^j \in \partial Q_{m_j+1} \setminus \{0\} \end{array} \right. \right\}. \tag{2.37}$$

2.4. Strong regularity and its relationship with constraint qualification and second-order optimality conditions. The notion of strong regularity has been introduced and investigated by Robinson in the landmark paper [24]. To the GE (2.32), it can be adapted as follows.

DEFINITION 13. *We say that the GE* (2.32) *is strongly regular at* (x^*, y^*) *(satisfies the strong regularity condition (SRC) at* (x^*, y^*)*), provided the (partially linearized) mapping* $\Sigma[\mathbb{R}^n \times \Pi_{j=1}^J \mathbb{R}^{m_j+1} \rightrightarrows \mathbb{R}^n \times \Pi_{j=1}^J \mathbb{R}^{m_j+1}]$ *defined by*

$$\Sigma(\delta_1, \delta_2) = \{(x, y) |\, \delta_1 = D^2_{xx} L(x^*, y^*)(x - x^*) - Dg(x^*)^\top (y - y^*),$$
$$\delta_2 \in g(x^*) + Dg(x^*)(x - x^*) + N_{\mathcal{Q}}(y)\}$$

has a single-valued Lipschitz localization around $(0, 0, x^*, y^*)$*; i.e., there exist a neighborhood* $\mathcal{U}$ *of* 0 *in* $\mathbb{R}^n \times \Pi_{j=1}^J \mathbb{R}^{m_j+1}$ *and a neighborhood* $\mathcal{V}$ *of* (x^*, y^*) *such that the mapping*

$$\Delta \colon (\delta_1, \delta_2) \mapsto \Sigma(\delta_1, \delta_2) \cap \mathcal{V}$$

is single-valued and Lipschitz on $\mathcal{U}$ *and* $\Delta(0, 0) = \{(x^*, y^*)\}$.

From [10, Theorem 2C.2] we know that the strong regularity of (2.32) at $(0, 0, x^*, y^*)$ is equivalent to the existence of a single-valued Lipschitz localization of S^e around $(0, 0, x^*, y^*)$.

Already in [24] the author examined the relationship between strong regularity and the SSOSC in the context of a classical mathematical programming problem. In SOCP, this relationship has been analyzed in [5]. To present this result, we invoke first the notion of nondegeneracy from section 2.2.

DEFINITION 14. *Let* x^* *be a feasible point of SOCP. We say that* x^* *is nondegenerate if*

$$Dg(x^*)\mathbb{R}^n + \operatorname{lin}(T_{\mathcal{Q}}(g(x^*))) = \Pi_{j=1}^J \mathbb{R}^{m_j+1}. \tag{2.38}$$

Observe that (2.38) amounts exactly to condition (2.18) with $K = \mathcal{Q}$.

One can easily see that x^* is nondegenerate whenever $Dg(x^*)$ is surjective. On the other hand, from (2.30), nondegeneracy directly implies *Robinson's CQ* condition (2.26) and, consequently, the existence of a Lagrange multiplier when x^* is a local minimizer of SOCP. Moreover, in this case nondegeneracy also implies the uniqueness of this multiplier.

To introduce the SSOSC in SOCP, we define first the $n \times n$ matrix $\mathcal{H}(x, y)$ as $\mathcal{H}(x, y) = \sum_{j=1}^J \mathcal{H}^j(x, y^j)$, where, for $j = 1, \ldots J$, we set

$$\mathcal{H}^j(x, y^j) := \begin{cases} -\frac{y_0^j}{s_0^j} Dg^j(x)^\top R_{m_j} Dg^j(x) & \text{if } s^j \in \partial Q_{m+1} \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.39}$$

In (2.39) one has (as defined above) $s^j = g^j(x)$, $j = 1, \dots J$, and

$$R_{m_j} := \begin{pmatrix} 1 & 0^\top \\ 0 & -I_{m_j} \end{pmatrix}.$$

Definition 15. (cf. [4, equation (3.20)] and [5, equation (46)].) *Let x^* be a critical point of SOCP and $y^* \in \Lambda(x^*)$. We say that the* second-order sufficient condition (SOSC) *holds at (x^*, y^*), provided*

$$Q_0(h) := h^\top D^2_{xx} L(x^*, y^*)h + h^\top \mathcal{H}(x^*, y^*)h > 0 \quad \forall h \in C(x^*) \setminus \{0\}. \tag{2.40}$$

We say that the SSOSC *holds at (x^*, y^*), provided*

$$Q_0(h) > 0 \quad \forall h \in \mathrm{Sp}(C(x^*)) \setminus \{0\}. \tag{2.41}$$

On the basis of (2.37), when x^* is nondegenerate, we easily derive that

$$\mathrm{Sp}(C(x^*)) = \left\{ h \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} d^j(h) = 0 & \text{if } y^{*j} \in \operatorname{int} Q_{m_j+1} \\ d^j(h) \in \mathbb{R}(y_0^{*j}, -\bar{y}^{*j}) & \text{if } y^{*j} \in \partial Q_{m_j+1} \setminus \{0\},\ s^j = 0 \\ \langle d^j(h), y^{*j} \rangle = 0 & \text{if } y^{*j},\ s^j \in \partial Q_{m_j+1} \setminus \{0\} \end{array} \right\}. \tag{2.42}$$

In particular, there is no condition on $d^j(h)$ if $y^{*j} = 0$.

The main result of Bonnans and Ramírez [5, Theorem 30] can now be stated as follows.

Theorem 16. *Let x^* be a local solution of problem SOCP and $y^* \in \Lambda(x^*)$. Then the GE* (2.32) *(KKT conditions) is strongly regular at (x^*, y^*) if and only if x^* is nondegenerate (Definition* 14*) and SSOSC holds at (x^*, y^*).*

In a recent paper [28], the previous characterization was extended in the following way.

Theorem 17. *Let x^* be a local solution of the problem SOCP satisfying Robinson's CQ condition* (2.30)*. Consider $y^* \in \Lambda(x^*)$. Then, the following statements are equivalent:*

(a) *x^* is nondegenerate and fulfills SSOSC at (x^*, y^*);*

(b) *the GE* (2.32) *is strongly regular at (x^*, y^*);*

(c) *any matrix of the form*

$$\begin{bmatrix} D^2_{xx} L(x^*, y^*) & -(Dg(x^*))^\top \\ (I - V)Dg(x^*) & V \end{bmatrix} \quad \text{with } V \in \operatorname{conv}\bar{\partial}_B P(g(x^*) - y^*) \tag{2.43}$$

is nonsingular.

This statement is strongly related to the results in the article of Sun [27], developed in a nonlinear semidefinite programming context; see also [6].

Besides these crucial results, we will also make use of the second-order necessary optimality conditions stated below.

Theorem 18. *Let the assumptions of Theorem* 17 *be fulfilled. Then, it holds that*

$$\sup_{y \in \Lambda(x^*)} h^\top D^2_{xx} L(x^*, y)h + h^\top \mathcal{H}(x^*, y)h \geq 0 \quad \forall h \in C(x^*). \tag{2.44}$$

Proof. See, for instance, [7], [5]. □

3. Aubin property and SSOSC. In this section we prove the main result of this paper: the characterization of the Aubin property of the critical point map S in terms of the SSOSC condition.

We limit first our attention to the problem SOCP with $J = 1$; i.e.,

$$\operatorname*{Min}_x f(x); \quad g(x) \in Q_{m+1}. \tag{3.1}$$

Hence,

$$S(u) = \{x | u \in Df(x) + (Dg(x))^\top N_{Q_{m+1}}(g(x))\}. \tag{3.2}$$

The first aim of this section is to characterize the Aubin property of S around $(0, x^*)$, where x^* is a (local) solution of (3.1). To this purpose we utilize the Mordukhovich criterion (2.9) and employ the following auxiliary statement. We recall that P denotes the metric projection onto Q_{m+1}.

LEMMA 19. *Let* $(\bar{a}, \bar{c}) \in \mathrm{Gr} N_{Q_{m+1}}$. *Then*

$$p \in D^* N_{Q_{m+1}}(\bar{a}, \bar{c})(q) \quad \textit{if and only if} \quad -q \in D^* P(\bar{a} + \bar{c}, \bar{a})(-q - p).$$

Proof. Since the well-known projection theorem (e.g., [10, p. 63]), we clearly have

$$\mathrm{Gr} N_{Q_{m+1}} = \left\{ (a, c) \middle| \begin{bmatrix} a + c \\ a \end{bmatrix} \in \mathrm{Gr} P \right\}.$$

Hence one has by virtue of [20, Theorem 1.17]

$$N_{\mathrm{Gr} N_{Q_{m+1}}}(\bar{a}, \bar{c}) = \{(p, r) | p = u + w, r = u, (u, w) \in N_{\mathrm{Gr} P}(\bar{a} + \bar{c}, \bar{a})\},$$

and the result follows from the definition of the coderivative. $\square$

Throughout the following we will assume that x^* is nondegenerate (Definition 14). This implies in particular the Robinson CQ condition. In the following result we establish a workable rule for verification of the Aubin property.

THEOREM 20. *Assume that x^* is nondegenerate. Then the multifunction S, given by* (3.2)*, has the Aubin property around* $(0, x^*)$ *if and only if in any solution pair* (v, b) *of the relations*

$$0 = D^2_{xx} L(x^*, y^*)v + (Dg(x^*))^\top (b - Dg(x^*)v), \tag{3.3a}$$

$$-Dg(x^*)v \in D^* P(g(x^*) - y^*, g(x^*))(-b) \tag{3.3b}$$

one has $v = 0$.

Proof. In order to compute the coderivative of S, we first note that

$$S(u) = \left\{ x \middle| \begin{bmatrix} x \\ u - Df(x) \end{bmatrix} \in \mathrm{Gr} N_\Gamma \right\}, \qquad \text{where } \Gamma = g^{-1}(Q_{m+1}).$$

$\mathrm{Gr} S$ is thus the preimage of $\mathrm{Gr} N_\Gamma$ in the mapping

$$\mathbf{\Phi}(u, x) = \begin{bmatrix} x \\ u - Df(x) \end{bmatrix}.$$

Since

$$D\Phi(0, x^*) = \begin{bmatrix} 0 & I_n \\ I_n & -D^2 f(x^*) \end{bmatrix}$$

is surjective, we can invoke [25, Exercise 6.7] and conclude that for all $z \in \mathbb{R}^n$,

$$D^* S(0, x^*)(z) = -\{v \in \mathbb{R}^n | 0 \in z + D^2 f(x^*)v + D^* N_\Gamma(x^*, -Df(x^*))(v)\}.$$

On the other hand, since the nondegeneracy assumption is fulfilled, the coderivative $D^* N_\Gamma$ can be computed via (2.21) of Theorem 7, which yields

$$D^* S(0, x^*)(z) = -\{v \in \mathbb{R}^n | 0 \in z + D^2_{xx} L(x^*, y^*)v + (Dg(x^*))^\top D^* N_{Q_{m+1}}(g(x^*), -y^*)(Dg(x^*)v)\}$$

for all $z \in \mathbb{R}^n$. On the basis of the Mordukhovich criterion we now infer that S possesses the Aubin property around $(0, x^*)$ if and only if the GE

$$0 \in D^2_{xx} L(x^*, y^*)v + (Dg(x^*))^\top D^* N_{Q_{m+1}}(g(x^*), -y^*)(Dg(x^*)v)$$

has only the trivial solution $v = 0$. So, it only remains to apply Lemma 19 with $p = b - Dg(x^*)v$, $q = Dg(x^*)v$ (consequently $-b = -p - q$), and relations (3.3a) follow. ∎

On the basis of (3.3a) and Theorems 5 and 6, we are now in position to state our main result of this section.

THEOREM 21. *Let x^* be a local solution of the problem SOCP, and let y^* be a corresponding Lagrange multiplier. Then, the following assertions are equivalent:*

(i) *x^* is nondegenerate (Definition 14) and SOCP fulfills the SSOSC (2.41) at (x^*, y^*);*

(ii) *the GE (2.32) (KKT conditions) is strongly regular at (x^*, y^*);*

(iii) *x^* is nondegenerate, and S has the Aubin property around $(0, x^*)$;*

(iv) *x^* is nondegenerate, and in any solution pair (v^*, b^*) of (3.3a) one has $v^* = 0$.*

Proof. Equivalence between statements (i) and (ii) has been established in [5, Theorem 30]. From statement (ii) it follows in particular that the mapping S^e has a single-valued Lipschitz localization around $(0, (x^*, y^*))$. Consequently, S has a single-valued Lipschitz localization around $(0, x^*)$ by virtue of (2.34). It follows that S possesses the (less stringent) Aubin property around $(0, x^*)$. Statements (iii) and (iv) are equivalent by virtue of Theorem 20 and so it remains to prove the implication

$$\text{(iv)} \Rightarrow \text{(i)}.$$

This statement is proved by contradiction. We split the proof according to the position of $g(x^*)$ and y^* in Q_{m+1} into six different cases:

Case 1. $y^* \in \operatorname{int} Q_{m+1}, g(x^*) = 0$. Since second-order necessary condition (2.44) holds at x^*, it is easy to see that SSOSC (2.41) is violated if and only if there is a nonzero vector $h \in \mathbb{R}^n$ such that

$$Dg(x^*)h = 0 \quad \text{and} \quad D^2_{xx} L(x^*, y^*)h = 0.$$

Indeed, it suffices to consider the standard spectral decomposition

$$D^2_{xx} L(x^*, y^*) = \sum \lambda_i q_i q_i^\top.$$

Necessary condition (2.44) amounts exactly to condition $\langle q_i, h\rangle = 0$ for all $i \in \{\{1, 2, \dots, n\} | \lambda_i < 0\}$ and all $h \in \operatorname{Ker} Dg(x^*)$. Consequently, if SSOSC (2.41) is violated, say by a nonzero vector $\bar{h}$, it follows that $\langle q_i, \bar{h}\rangle = 0$ for all i. This argumentation will also be used in the remaining cases, possibly with other matrices, different from $D^2_{xx}L(x^*, y^*)$.

In this case, it is easy to check that $v = h$ and $b = 0$ satisfy (3.3a) and, since $D^*P(-y^*, 0)(0) = \{0\}$, they also satisfy (3.3b). We thus obtain a solution (v, b) of (3.3a) satisfying $v \neq 0$.

Case 2. $g(x^*) \in \operatorname{int} Q_{m+1}$ and $y^* = 0$. The second-order necessary condition (2.44) is equivalent to positive semidefiniteness of the Hessian $D^2_{xx}L(x^*, y^*)$. Thus, in this case, (2.41) is violated if and only if there is a nonzero vector $h \in \mathbb{R}^n$ such that $D^2_{xx}L(x^*, y^*)h = 0$. Hence, since $I_{m+1} \in D^*P(g(x^*), g(x^*))$, vectors $v = h = h - D^2_{xx}L(x^*, y^*)h$ and $b = Dg(x^*)h$ are a solution of (3.3a) satisfying $v \neq 0$.

To simplify the notation, from now on we set $s := g(x^*)$ and $d := Dg(x^*)h$.

Case 3. $g(x^*), y^* \in \partial Q_{m+1} \setminus \{0\}$. In this case, (2.41) does not hold if and only if there is a nonzero vector h such that $\langle d, y^*\rangle = 0$ and

$$Q_0(h) = h^\top D^2_{xx}L(x^*, y^*)h - \alpha h^\top Dg(x^*)^\top R Dg(x^*)h \le 0,$$

where $\alpha := \frac{y_0^*}{s_0} (> 0)$ and we have set $R := R_{m+1} = \begin{pmatrix} 1 & 0^\top \\ 0 & -I_m \end{pmatrix}$. This together with second-order necessary optimality condition (2.44) (which says that the quadratic function $Q_0(h)$ is nonnegative over the linear space $(y^*)^\perp$) yields

$$D^2_{xx}L(x^*, y^*)h = \frac{y_0^*}{s_0} Dg(x^*)^\top R Dg(x^*)h. \tag{3.4}$$

We claim that the vectors $v = h$ and $b = (I_{m+1} - \alpha R)d$ satisfy (3.3a). Indeed, by replacing v by h and using equality (3.4), the right-hand side of (3.3a) is written as

$$Dg(x^*)^\top[\alpha R d + b - d],$$

which is equal to 0 if and only if $b = (I_{m+1} - \alpha R)d$.

In order to prove (3.3b), we need only prove that d solves

$$d = A(z)(I_{m+1} - \alpha R)d, \tag{3.5}$$

where $z = g(x^*) - y^*$, and

$$A(z) = \frac{1}{2}\begin{bmatrix} 1 & \bar{w}^\top \\ \bar{w} & H \end{bmatrix}$$

with $\bar{w} = \frac{\bar{z}}{\|\bar{z}\|}$, $H = (1 + \frac{z_0}{\|\bar{z}\|})I_m - \frac{z_0}{\|\bar{z}\|}\bar{w}\bar{w}^\top$. This is a particular form of inclusion (3.3b) in this case. To prove the above claim, observe that

$$(I_{m+1} - \alpha R)d = \begin{bmatrix} (1-\alpha)d_0 \\ (1+\alpha)\bar{d} \end{bmatrix}.$$

From the orthogonality of $s = g(x^*)$ and y^*, it follows that $s_0(x^*)y_0^* + \langle \bar{s}, \bar{y}^*\rangle = 0$, and

$$\|\bar{z}\| = \sqrt{\|\bar{s}\|^2 + \|\bar{y}^*\|^2 - 2\langle \bar{s}, \bar{y}^*\rangle} = \sqrt{(s_0)^2 + (y_0^*)^2 + 2s_0 y_0^*} = s_0 + y_0^*. \tag{3.6}$$

The first row in (3.5) attains the form

$$2d_0 = (1-\alpha)d_0 + (1+\alpha)\langle \bar{w}, \bar{d}\rangle,$$

which reduces to

$$d_0 = \langle \bar{w}, \bar{d}\rangle. \tag{3.7}$$

This is equivalent to the orthogonality condition

$$\left\langle d, \begin{bmatrix} 1 \\ -\bar{w} \end{bmatrix} \right\rangle = 0. \tag{3.8}$$

By virtue of (3.6),

$$\left\langle d, \begin{bmatrix} 1 \\ -\bar{w} \end{bmatrix} \right\rangle = \frac{1}{s_0 + y_0^*}[d_0(s_0 + y_0^*) - \langle \bar{d}, \bar{s} - \bar{y}^*\rangle] = \frac{1}{s_0 + y_0^*}[d_0 s_0 + \langle \bar{d}, -\bar{s}\rangle],$$

because $\langle y^*, d\rangle = 0$. Moreover, since the vector $(s_0, -\bar{s})$ is orthogonal to $s = g(x^*)$ and belongs to $\partial Q_{m+1} \setminus \{0\}$, it is colinear with y^* (see (2.4)) and equality (3.8) holds true.

The "second" row of the right-hand side of (3.5) attains the form

$$\frac{1}{2}\left[\frac{s_0 - y_0^*}{s_0} d_0 \bar{w} + 2\bar{d} - \frac{s_0 - y_0^*}{s_0} \bar{w}\bar{w}^\top \bar{d}\right].$$

From (3.7) it follows that $d_0\bar{w} = \bar{w}\bar{w}^\top\bar{d}$ and so the whole above expression reduces to $\bar{d}$. Consequently, we have found a solution (v, b) of (3.3a) such that $v \neq 0$, which contradicts (iv).

Case 4. $y^* \in \partial Q_{m+1} \setminus \{0\}$, $g(x^*) = 0$. In this case, (2.41) does not hold if and only if there is a nonzero vector $h \in \mathbb{R}^n$ such that

$$Dg(x^*)h \in \mathbb{R}Ry^* \quad \text{and} \quad D^2_{xx}L(x^*, y^*)h = 0. \tag{3.9}$$

Indeed, note that in this case the second-order necessary condition (2.44)

$$h^\top D^2_{xx}L(x^*, y^*)h \geq 0 \quad \forall h\colon Dg(x^*)h \in \mathbb{R}_+Ry^*$$

is actually equivalent to

$$h^\top D^2_{xx}L(x^*, y^*)h \geq 0 \quad \forall h\colon Dg(x^*)h \in \mathbb{R}Ry^*,$$

deducing that if (2.41) does not hold, then there exists $h \neq 0$ satisfying (3.9). The equivalence follows.

Now, from the spectral decomposition of y^*, it follows that

$$y^* \in \mathbb{R}c_2(y^*) = \mathbb{R}\begin{bmatrix} 1 \\ \frac{\bar{y}^*}{\|\bar{y}^*\|} \end{bmatrix} \quad \text{so that } Ry^* \in \mathbb{R}\begin{bmatrix} 1 \\ -\frac{\bar{y}^*}{\|\bar{y}^*\|} \end{bmatrix} = \mathbb{R}c_2(-y^*)$$

and from Theorem 5 (ii) it follows that for any $u^* \in \mathbb{R}^{m+1}$

$$u^* - \mathrm{Proj}_{(c_2(-y^*))^\perp}(u^*) \in D^*P(-y^*, 0)(u^*).$$

Hence, with $u^* = -Dg(x^*)h$, we conclude that the element

$$-Dg(x^*)h - \mathrm{Proj}_{(c_2(-y^*))^\perp}(-Dg(x^*)h) = -Dg(x^*)h$$

definitely belongs to the coderivative $D^*P(-y^*, 0)(-Dg(x^*)h)$. It follows that (3.3a) has a solution

$$v = h \neq 0, \qquad b = Dg(x^*)h.$$

Case 5. $g(x^*) \in \partial Q_{m+1} \setminus \{0\}$, $y^* = 0$. The second-order necessary condition (2.44) attains the form

$$h^\top D^2_{xx}L(x^*, y^*)h \geq 0 \quad \forall h\colon g(x^*)^\top RDg(x^*)h \geq 0,$$

which is equivalent to positive semidefiniteness of the Hessian $D^2_{xx}L(x^*, y^*)$. Indeed, for any vector h such that the real number $g(x^*)^\top RDg(x^*)h$ is negative, it suffices to take $-h$ instead of h in the above relation, and the positive semidefiniteness of $D^2_{xx}L(x^*, y^*)$ follows directly. Thus, in this case, (2.41) is violated if and only if there is a nonzero vector $h \in \mathbb{R}^n$ such that $D^2_{xx}L(x^*, y^*)h = 0$. Hence, since $I_{m+1} \in D^*P(g(x^*), g(x^*))$, vectors $v = h$ and $b = Dg(x^*)h$ are a solution of (3.3) satisfying $v \neq 0$.

Case 6. $y^* = g(x^*) = 0$. In this case $C(x^*) = \{h \in \mathbb{R}^n\colon Dg(x^*)h \in Q_{m+1}\} = \{h \in \mathbb{R}^n | d_0 \geq \|\bar{d}\|\}$. Consequently, we have

$$\begin{aligned}(Dg(x^*))^{-1}(Q_{m+1} \cup -Q_{m+1}) &= \{h \in \mathbb{R}^n | d_0^2 \geq \|\bar{d}\|^2\} \\ &= \{h \in \mathbb{R}^n | h^\top Dg(x^*)^\top RDg(x^*)h \geq 0\},\end{aligned}$$

and then the second-order necessary condition attains the form

$$h^\top D^2_{xx}L(x^*, y^*)h \geq 0 \quad \forall h\colon h^\top Dg(x^*)^\top RDg(x^*)h \geq 0. \tag{3.10}$$

On the other hand, since in this case $\mathrm{Sp}(C(x^*)) = \mathbb{R}^n$, we infer that the SSOSC (2.41) is violated if and only if $D^2_{xx}L(x^*, y^*)$ is not positive definite. Then, thanks to S-lemma (see, e.g., [3, Theorem 4.3.3]), we know that (3.10) is equivalent to saying that there exists $\gamma > 0$ such that

$$D^2_{xx}L(x^*, y^*) - \gamma Dg(x^*)^\top RDg(x^*) \succcurlyeq 0. \tag{3.11}$$

We split here our proof into two subcases:

(a) Let us suppose that (2.41) is violated due to the existence of a nonzero vector h satisfying $h^\top D^2_{xx}L(x^*, y^*)h < 0$. Then we can reduce γ (if necessary) until the matrix $D^2_{xx}L(x^*, y^*) - \gamma Dg(x^*)^\top RDg(x^*)$ loses its positive definiteness but not its positive semidefiniteness (note this is always possible due to the continuity of the eigenvalues). Hence, in its kernel, there exists a nonzero vector which is, for the sake of simplicity, also denoted by h. We thus obtain

$$D^2_{xx}L(x^*, y^*)h = \gamma Dg(x^*)^\top RDg(x^*)h.$$

We claim that the vector $-d = -Dg(x^*)h$ belongs to the set

$$\bar{\partial}_B P(0)(-d+\gamma R d) = \bar{\partial}_B P(0)\left(-\begin{bmatrix}(1-\gamma)d_0\\(1+\gamma)\bar{d}\end{bmatrix}\right)$$

(where $\bar{\partial}_B P(a)b$ stands for the set of elements of the form Ab with $A \in \bar{\partial}_B P(a)$). Indeed, it suffices to select in (2.15) a matrix specified by a unit vector w such that $d^\top(1,-w)=0$ and $\alpha = 1/(1+\gamma)$. Note that the existence of such w is ensured due to inequality $h^\top D^2_{xx}L(x^*,y^*)h < 0$ which, together with (3.11), implies that $\|\bar{d}\|^2 > d_0^2$ or, equivalently, $\|\bar{d}\| > |d_0|$. This condition clearly ensures the existence[1] of a unit vector w such that $\langle \bar{d}, w\rangle = d_0$. Now, since $D^*P(0)u^*$ contains $\bar{\partial}_B P(0)u^*$ for all u^* (see (2.14)), we conclude that $-d$ belongs to $D^*P(0)(-d+\gamma Rd)$. Our claim is proven.

Finally, it is easy to see that relations (3.3a) are solved by the vectors $v = h$ and $b = d - \gamma R d = (I_{m+1} - \gamma R)Dg(x^*)h$. This contradicts the statement (iv).

(b) Otherwise, if this situation is due to the existence of a nonzero vector h satisfying $h^\top D^2_{xx}L(x^*,y^*)h = 0$ (so that $D^2_{xx}L(x^*,y^*)$ is positive semidefinite), it necessarily follows that

$$D^2_{xx}L(x^*,y^*)h = 0.$$

Here, from the fact that $I_{m+1} \in \bar{\partial}_B P(0)$, the vector $-d = -Dg(x^*)h$ trivially belongs to the set

$$\bar{\partial}_B P(0)(-d).$$

Now, since $D^*P(0)u^*$ contains $\bar{\partial}_B P(0)u^*$ for all u^* (see (2.14)), we conclude that $-d$ belongs to $D^*P(0)(-d)$ in this case. We thus deduce that relations (3.3a) are solved by $v = h$ and $b = Dg(x^*)h$, arriving at a contradiction with the statement (iv). □

Remark 22. As pointed out by one of the reviewers, the statement of Theorem 21 can also be derived directly on the basis of available results from standard nonlinear programming whenever $g_0(x^*) > 0$. Then, namely, under the imposed assumptions,

$$N_{g^{-1}(Q_{m+1})}(x^*) = (Dg(x^*))^\top \begin{bmatrix}-2g_0(x^*)\\ 2g_1(x^*)\\ \vdots \\ 2g_m(x^*)\end{bmatrix} N_{\mathbb{R}_-}(h \circ g(x^*)),$$

where $h(y) := \bar{y}^2 - y_0^2$. Moreover $\nabla(h\circ g)(x^*)$ is surjective and so one can employ the results in [9] concerning variational inequalities with polyhedral convex sets. When $g_0(x^*) = 0$, however, this direct way is not passable.

Remark 23. The previous proof could be much shortened by using the results of [28] (see Theorem 17). Indeed, by comparing Theorem 17, condition (c) and Theorem 20, the proof of Cases 1–5 is straightforward, and so it remains only to prove (the most difficult)

[1]Notice that this argument fails when $m = 1$, i.e., when w is a real value. However, in this case, constraint $g(x) \in Q_2$ can be rewritten as two nonlinear constraints $h_i(x) \le 0$, $i = 1, 2$, where $h_i(x) := g_0(x) + (-1)^i g_1(x)$. In this context, nondegeneracy (Definition 14) coincides with the linear independence of $\nabla h_1(x^*)$ and $\nabla h_2(x^*)$ (both functions are clearly active at x^*), and consequently, known results on nonlinear programming (e.g., [9]) give us the desired implication (iv) ⇒ (i).

Case 6 (when $y^* = g(x^*) = 0$). However, the arguments used in the above proof are necessary in order to extend our main result to the several cone framework conducted in Theorem 26.

4. Extension to several second-order cones.

In this section we intend to extend Theorem 21 to the general SOCP problem

$$\operatorname*{Min}_{x\in\mathbb{R}^n} f(x); \quad g^j(x) \in Q_{m_j+1}, \quad j = 1, 2, \ldots, J.$$

By following the same arguments as in Lemma 19 and Theorem 20 we can prove the following equivalence.

THEOREM 24. *Assume that x^* is nondegenerate. Then the multifunction S, given by* (3.2)*, has the Aubin property around* $(0, x^*)$ *if and only if the equality* $v = 0$ *holds for any solution pair* (v, b) *of the relations*

$$0 = D^2_{xx}L(x^*, y^*)v + (Dg(x^*))^\top(b - Dg(x^*)v), \tag{4.1a}$$

$$-\,Dg(x^*)v \in D^*P(g(x^*) - y^*, g(x^*))(-b), \tag{4.1b}$$

where P now stands for the projection operator onto the Cartesian product of second-order cones $\mathcal{Q} = \Pi_{j=1}^J Q_{m_j+1}$.

Proof. It follows from Lemma 19 and the fact that the limiting normal-cone to the Cartesian product of a finite number of sets amounts to the Cartesian product of the normal-cones to the single sets; cf. [25, Proposition 6.41]. ∎

Remark 25. Note that condition (4.1b) can be written in a product form

$$-Dg^j(x^*)v \in D^*P^j(g^j(x^*) - y^{*j}, g^j(x^*))(-b^j), \qquad j = 1, \ldots, J,$$

where P^j is the projection operator onto the second-order cone Q_{m_j+1}.

On the basis of (4.1a) and Theorems 5 and 6, we are now in position to state our main result.

THEOREM 26. *Let x^* be a local solution of the problem SOCP, and let y^* be a corresponding Lagrange multiplier. Suppose that there is at most one block j such that either $g^j(x^*) = 0$ and $y^{*j} \in \partial Q_{m_j+1} \setminus \{0\}$ or $g^j(x^*) \in \partial Q_{m_j+1} \setminus \{0\}$ and $y^{*j} = 0$ or $g^j(x^*) = 0 = y^{*j}$. Then, the following assertions are equivalent:*

(i) *x^* is nondegenerate (Definition* 14*) and SOCP fulfills the SSOSC* (2.41) *at* (x^*, y^*)*;*

(ii) *the GE* (2.32) *(KKT system) is strongly regular at* (x^*, y^*)*;*

(iii) *x^* is nondegenerate, and S has the Aubin property around* $(0, x^*)$*;*

(iv) *x^* is nondegenerate, and in any solution pair* (v^*, b^*) *of* (4.1a) *one has* $v^* = 0$.

Proof. Since the results established in [5] and [22] are valid for the Cartesian product of several second-order cones $\mathcal{Q} = \Pi_{j=1}^J Q_{m_j+1}$, we can argue exactly as at the beginning of the proof of Theorem 21. So, only the implication

$$\text{(iv)} \Rightarrow \text{(i)}$$

needs to be demonstrated. This statement is also proved by contradiction and the proof uses the same approach as the proof of Theorem 21, where only one cone has been considered. However, the combination of different cases (corresponding to different positions of $g^j(x^*)$ and y^{*j} in Q_{m_j+1}) is not straightforward and deserves to be explained in detail. For this, let us denote by J_i, with $i = 1, \ldots, 6$, the sets of indexes corresponding to cases 1 to 6 in the proof of Theorem 21. So, by our assumptions, we know that $|J_4 \cup J_5 \cup J_6| \leq 1$.

For the sake of simplicity, we employ again the notation: $s^j := g^j(x^*)$ and $d^j(h) := Dg^j(x^*)h$.

Note that $h^\top \mathcal{H}^j(x^*, y^{*j})h \neq 0$ only when $j \in J_3$. Consequently,

$$h^\top \mathcal{H}(x^*, y^*)h = \sum_{j \in J_3} h^\top \mathcal{H}^j(x^*, y^{*j})h = -\sum_{j \in J_3} \frac{y_0^{*j}}{s_0^j} d^j(h)^\top R_{m_j} d^j(h).$$

Then, the quadratic form $Q_0(\cdot)$, appearing in SSOSC (2.41), is given by

$$Q_0(h) := h^\top D_{xx}^2 L(x^*, y^*)h - \sum_{j \in J_3} \frac{y_0^{*j}}{s_0^j} d^j(h)^\top R_{m_j} d^j(h).$$

Second-order necessary optimality condition (2.44) says that $Q_0(h)$ is nonnegative over the critical cone $C(x^*)$, and consequently, it is also nonnegative over $C(x^*) \cup -C(x^*)$. But, since $|J_4 \cup J_5 \cup J_6| \leq 1$, the latter set is exactly equal to

$$\tilde{C}(x^*) := \{h \in \mathbb{R}^n : d^j(h) \in \pm T_{Q_{m_j+1}}(s^j) \cap (y^{*j})^\perp \quad \forall j = 1, \ldots, J\}.$$

Assume now that $J_6 = \varnothing$ so that we have to do with either $J_4 \cup J_5 \cup J_6 = \varnothing$, or with a unique $j \in J_4 \cup J_5$. In both cases one has

$$\tilde{C}(x^*) = \mathrm{Sp}(C(x^*)).$$

This implies that (2.41) is violated if and only if $Q_0(h) = 0$ for some $h \in \mathrm{Sp}(C(x^*)) \setminus \{0\}$, or, equivalently, if and only if there exists a nonzero vector $h \in \mathrm{Sp}(C(x^*))$ such that

$$D_{xx}^2 L(x^*, y^*)h + \mathcal{H}(x^*, y^*)h = 0. \tag{4.2}$$

The proof can now be finished on the basis of arguments provided below after formula (4.5).

To deal with the situation when $|J_6| = 1$, we introduce a subspace E of $\mathbb{R}^n$ defined by

$$E := \left\{ h \in \mathbb{R}^n \,\middle|\, \begin{array}{ll} d^j(h) = 0 & \text{if } y^j \in \operatorname{int} Q_{m_j+1} \\ \langle d^j(h), y^j \rangle = 0 & \text{if } y^j \in \partial Q_{m_j+1} \setminus \{0\} \end{array}, \ j = 1, 2, \ldots, J \right\} \tag{4.3}$$

(there is no restriction on $d^j(h)$ if $y^j = 0$). Clearly, one has the inclusions

$$\tilde{C}(x^*) \subseteq \mathrm{Sp}(C(x^*)) \subseteq E.$$

Moreover, we observe that for $j \in J_6$ the condition $h \in \tilde{C}(x^*)$ amounts to $h \in C(x^*)$ and the nonnegativity of the quadratic form

$$Q_1(h) := d^j(h)^\top R_{m_j} d^j(h) = (d_0^j(h))^2 - \|\bar{d}^j(h)\|^2.$$

It follows that

$$\tilde{C}(x^*) = \mathrm{Sp}(C(x^*)) \cap Q_1^{-1}(\mathbb{R}_+) = E \cap Q_1^{-1}(\mathbb{R}_+).$$

Thanks to a slightly improved version of the S-lemma (e.g., [23, Proposition 3.9]), we now infer that there exists a real $\gamma > 0$ such that

$$Q_0(h) - \gamma Q_1(h) \geq 0 \quad \forall h \in E. \tag{4.4}$$

Suppose now the existence of $h \in \mathrm{Sp}(C(x^*)) \setminus \{0\}$ such that (2.41) does not hold; that is, $Q_0(h) \leq 0$. Now, following the arguments given in Case 6 of Theorem 21, we split our proof into two subcases:

(a) If (2.41) is violated because $Q_0(h) < 0$, then (4.4) yields $Q_1(h) < 0$. In this case, without loss of generality, we can assume (reducing γ if necessary) that $Q_0(h) - \gamma Q_1(h) = 0$. So, we obtain from (4.4) that

$$D^2_{xx}L(x^*, y^*)h + \mathcal{H}(x^*, y^*)h = \gamma Dg^j(x^*)^\top R_{m_j} Dg^j(x^*)h.$$

It can be proven that $-d = -d^j(h) = -Dg^j(x^*)h$ belongs to the set

$$\bar{\partial}_B P^j(0)(-d + \gamma R_{m_j} d) = \bar{\partial}_B P^j(0)\left(-\begin{bmatrix}(1-\gamma)d_0 \\ (1+\gamma)\bar{d}\end{bmatrix}\right).$$

Indeed, it suffices to choose a unit vector w such that $d^\top(1, -w) = 0$ and $\alpha = 1/(1+\gamma)$ in (2.15). As in the proof of Case 6 of Theorem 21, the existence of such w is ensured due to inequality $Q_0(h) < 0$ which, together with (4.4), implies that $\|\bar{d}\|^2 > d_0^2$ or, equivalently, $\|\bar{d}\| > |d_0|$. This condition clearly ensures the existence of a unit vector w such that $\langle \bar{d}, w\rangle = d_0$. For the case when $m_j = 1$, see Remark 28 below.

Now, since $D^*P^j(0)(u^*)$ contains $\bar{\partial}_B P^j(0)(u^*)$ for all u^* (see (2.14)), we conclude that $-d$ belongs to $D^*P^j(0)(-d + \gamma R_{m_j} d)$. Consequently, (v, b^j) with $v = h$ and $b^j = d - \gamma R_{m_j} d = (I_{m_j+1} - \gamma R_{m_j})Dg^j(x^*)h$ solves (4.1a) for the block $j \in J_6$.

(b) Otherwise, if this situation is due to the existence of a nonzero vector h satisfying $Q_0(h) = 0$ (so that $Q_0(\cdot)$ is nonnegative), it necessarily follows that

$$D^2_{xx}L(x^*, y^*)h + \mathcal{H}(x^*, y^*)h = 0. \tag{4.5}$$

Then, from the fact that $I_{m_j+1} \in \bar{\partial}_B P^j(0)$, it trivially holds that vector $-d = -d^j(h) = -Dg^j(x^*)h$ belongs to

$$\bar{\partial}_B P^j(0)(-d).$$

Once again, since $D^*P^j(0)(u^*)$ contains $\bar{\partial}_B P^j(0)(u^*)$ for all u^*, we obtain that $-d$ belongs to $D^*P^j(0)(-d)$. We thus conclude that the solution of (4.1a), for the block $j \in J_6$, is $v = h$ and $b^j = Dg^j(x^*)h$.

For the other blocks j, the same arguments as in Theorem 21 ensure that the respective vectors b^j can be chosen as follows:

$$b^j = \begin{cases} 0 & \text{if } j \in J_1, \\ d^j(h)(= Dg^j(x^*)h) & \text{if } j \in J_2, \\ (I_{m_j+1} - \alpha_j R_{m_j})d^j(h) \quad \text{with } \alpha_j = y_0^{*j}/s_0^j & \text{if } j \in J_3. \end{cases}$$

If the unique index j belongs to $J_4 \cup J_5$, then, on the basis of (4.2) and the above discussion, one can put $v = h$ and $b^j = Dg^j(x^*)h$ (as in the case of J_2) and we obtain again a solution of (4.1a) with a nonzero vector v. The statement has been established. $\square$

Remark 27. If the index sets J_4, J_5, and J_6 are empty, one says that x^* is a *strict complementary solution* of SOCP. In this case, it is possible to prove the statement of Theorem 26 in a simpler way, by using the classical implicit function theorem. In fact, this statement then holds true even if $\mathcal{Q}$ is a general symmetric cone, provided the projection onto $\mathcal{Q}$ is continuously differentiable at $g(x^*) - y^*$.

Remark 28. In the case when J_4 and J_5 are empty, J_6 is a singleton and, for this index $j \in J_6$, one has $m_j = 1$; the statement of Theorem 26 follows from the previous remark and the fact that constraint $g_j(x) \in Q_2$, with $j \in J_6$, can be cast as classical nonlinear programming constraint (see footnote [1]).

Let us now illustrate the situation when $|J_4 \cup J_5 \cup J_6| > 1$ by means of the parametric SOCP consisting of minimizing the function

$$f_\xi(x^1, x^2) = \frac{1}{2}\|x^1 - a\|^2 + \frac{1}{2}\|x^2 - b\|^2 + \xi x_1^1 x_1^2 \quad (x^1, x^2 \in \mathbb{R}^3) \tag{4.6}$$

over the product of two second-order cones of dimension 3 (that is, $x = (x^1, x^2)$ is constrained to $Q_3 \times Q_3$). Here ξ is a positive parameter and we have set the vectors $a = (\xi, \xi/(1+\xi), 0)^\top$ and $b = (\xi, 0, \xi/(1+\xi))^\top$.

It can be easily proved that $x^* = (x^{*1}, x^{*2})$, with $x^{*1} = (\xi/(1+\xi), \xi/(1+\xi), 0)^\top$ and $x^{*2} = (\xi/(1+\xi), 0, \xi/(1+\xi))^\top$, is a critical point of this problem for each $\xi > 0$, with $y^* = 0$ being its corresponding unique Lagrange multiplier. Moreover, x^* (in fact every feasible point) is nondegenerate, because $g(\cdot)$ is the identity function and hence $Dg(x^*)$ is surjective. So, we have to do for all $\xi > 0$ with Case 5 of the preceding analysis (i.e., both blocks are in J_5). Let us first examine the Aubin property of the corresponding map S_ξ, where the subscript signalizes the dependence on ξ. Since $g(\cdot)$ is the identity, the Hessian of the Lagrangian coincides with the Hessian of f_ξ, which is given (for all x) by the block matrix

$$\nabla^2 f_\xi(x) = \begin{pmatrix} I_3 & C_\xi \\ C_\xi & I_3 \end{pmatrix}, \tag{4.7}$$

where

$$C_\xi = \begin{pmatrix} \xi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Moreover, condition (4.1a) attains a simpler form

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in (I_6 - \nabla^2 f_\xi(x^*)) \begin{pmatrix} D^*P^1(x^{*1})(v_1) \\ D^*P^2(x^{*2})(v_2) \end{pmatrix} \Rightarrow v_1 = v_2 = 0, \tag{4.8}$$

where P^1 and P^2 are both equal to the projection onto Q_3. From now on, this projection is simply denoted by P.

Thanks to the particular structure of the matrix (4.7), the relation of the left-hand side of (4.8) can be written as

$$v_1 \in C_\xi D^*P(x^{*2})(v_2) \quad \text{and} \quad v_2 \in C_\xi D^*P(x^{*1})(v_1).$$

Now, we observe that by part (i) of Theorem 5 each element $u \in D^*P(x^{*1})(v_1)$, $w \in D^*P(x^{*2})(v_2)$ can be expressed in the form

Table 4.1
Summary of the example.

ξ	x^* is a local minimum	Strong regularity	Aubin property
$0 \leq \xi < 1$	Yes	Yes	Yes
$1 \leq \xi < 2$	Yes	No	No
$\xi = 2$	No	No	No
$\xi > 2$	No	No	Yes

$$u = G(x^{*1})v_1, \quad w = H(x^{*2})v_2$$

with appropriate matrices $G(x^{*1})$, $H(x^{*2}) \in \mathbb{R}^{3\times 3}$ given by (2.12). This, along with the fact that $D^*P(x^{*1})(0) = D^*P(x^{*2})(0) = \{0\}$ (because P is Lipschitz), enables us to characterize the Aubin property of S_ξ around x^* by the simple condition

$$\xi^2 g_{11}(x^{*1})h_{11}(x^{*2}) \neq 1, \tag{4.9}$$

where $g_{11}(x^{*1})$ and $h_{11}(x^{*2})$ are the entries located in the position (1,1) of matrices $G(x^{*1})$ and $H(x^{*2})$, respectively. In this way we have proved the following statement: *The map S_ξ has the Aubin property around x^* if and only if either $\xi < 1$ or $\xi > 2$.*

The proof follows directly from (4.9) and the fact that, by (2.12), both entries $g_{11}(x^{*1})$ and $h_{11}(x^{*2})$ range in the interval [0.5,1].

Next we will analyze for what values of ξ the point x^* is a minimizer of our problem, and when the perturbed KKT system is strongly regular at (x^*, y^*). Since the KKT system is strongly regular at (x^*, y^*) if and only if the SSOSC (2.41) is fulfilled at (x^*, y^*), and since in this case $\mathrm{Sp}(C(x^*)) = \mathbb{R}^6$, this holds true if and only if the Hessian of f_ξ is positive definite. This is clearly equivalent to $\xi < 1$. Now, by using the characterization of $C(x^*)$ given in (2.37), we can verify that the second-order sufficient condition (2.40) is fulfilled if and only if $\xi < 2$, but the second-order necessary optimality condition (2.44) is fulfilled if and only if $\xi \leq 2$. It follows that a discrepancy between the Aubin property and the strong regularity appears only for $\xi > 2$, when x^* is not a local minimum of the analyzed SOCP. Finally, when $\xi = 2$, we can verify by simple inspection that x^* is not a local minimum of our problem. Indeed, for $x = ((0,0,0)^\top, (2,0,2/3)^\top)$ belonging to $Q_3 \times Q_3$, it is easy to check that the directional derivative of f_ξ at x^* in the direction $d = x - x^*$ is negative.

On the basis of the preceding analysis we display a summary for this example in Table 4.1.

5. Conclusions. This paper presents three new results associated with conical programming. First, Theorem 7 contains a new formula for the coderivative of the normal-cone mapping $N_A(\cdot)$, when A is a preimage of a convex set K in a smooth mapping $g(\cdot)$. This formula generalizes [21, Theorem 3.4] and has potential to be applied even when K is not a cone.

On the basis of this result and [22], where the limiting (Mordukhovich) coderivative of the metric projection onto the second-order cone has been computed, we have derived a workable condition characterizing the Aubin property of a canonically perturbed necessary optimality condition for SOCP under nondegeneracy (Theorem 24).

Our main result is Theorem 26, providing a relationship between the Aubin property and Robinson's strong regularity of the canonically perturbed KKT system associated with SOCP. For this, we use a characterization for the strong regularity property in terms of the SSOSC and the nondegeneracy condition given in [5] for SOCP. It is very close to the relationship between the Aubin property and the strong regularity of the canonically perturbed KKT system in the case of nonlinear programming. Unfortunately, the equivalences stated in Theorem 26 have been proved only under the assumption that, at the reference pair (x^*, y^*), among all blocks $j \in \{1, \ldots, J\}$ there is at most one violating the strict complementarity condition; i.e., there is at most one block j satisfying either $g^j(x^*) = 0$ and $y^{*j} \in \partial Q_{m_j+1} \setminus \{0\}$ or $g(x^*) \in \partial Q \setminus \{0\}$ and $y^{*j} = 0$ or $g^j(x^*) = 0 = y^{*j}$. It is currently not clear whether this restriction is due to the applied proof technique, based on the S-lemma, or whether it is principally related to the geometry of the second-order cones.

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