Shape optimization in 3D contact problems with Coulomb friction

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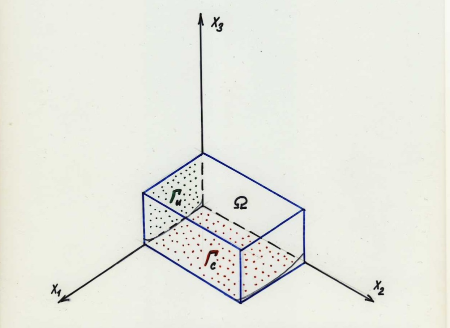
P. Beremlijski, J. Haslinger, M. Kočvara, R. Kučera and J.V. Outrata



Elastic body: $\Omega \subset \mathbb{R}^3$, $\mathrm{bd}\,\Omega = \Gamma_u \cup \Gamma_p \cup \Gamma_c$

Γ_u ... zero displacements

Γ_p ... surface tractions

Γ_c ... boundary in contact with the "rigid" plane $x_1 \times x_2$.

After a FE discretization one obtains a finite-dimensional Mathematical Program with Equilibrium Constraints (MPEC)

$$
\begin{aligned}
&\text{minimize } f(\alpha, y) \\
&\text{subject to} \quad y \in S(\alpha) \\
&\qquad\qquad\quad\ \alpha \in U_{ad}\,.
\end{aligned}
\tag{1}
$$

Introduction

In (1) the design variable $\alpha \in \mathbb{R}^p$ specifies the shape of the contact boundary and the state variable $y = (u_\tau, u_\nu, \lambda) \in \mathbb{R}^{2p} \times \mathbb{R}^p \times \mathbb{R}^p_+$, where

p ..... number of nodes in the contact boundary
u_τ ..... vector of tangential displacements
u_ν .... vector of normal displacements
λ ..... multiplier associated with the nonpenetrability constraint

$$u_\nu + \alpha \geqq 0.$$

Further, in (1) f is an objective, U_{ad} is the set of admissible design variables (controls) and the control-state map S is given by the generalized equation (GE)

$$\begin{aligned}
0 &\in A_{\tau\tau}(\alpha)u_\tau + A_{\tau\nu}(\alpha)u_\nu - \ell_\tau(\alpha) + \tilde{Q}(u_\tau, \lambda) \\
0 &= A_{\nu\tau}(\alpha)u_\tau + A_{\nu\nu}(\alpha)u_\nu - \ell_\nu(\alpha) \qquad (2) \\
0 &\in u_\nu + \alpha + N_{\mathbb{R}^p_+}(\lambda),
\end{aligned}$$

where $A_{\tau\tau}, A_{\tau\nu}, A_{\nu\tau}, A_{\nu\nu}$ are blocks of appropriate restriction of stiffness matrix,
ℓ_τ, ℓ_ν reflect the action of external forces

and

$$\tilde{Q}(u_\tau, \lambda) = \mathcal{F} \begin{bmatrix} \lambda^1 \partial \| u_\tau^1 \| \\ \lambda^2 \partial \| u_\tau^2 \| \\ \vdots \\ \lambda^p \partial \| u_\tau^p \| \end{bmatrix}$$

with $\mathcal{F} > 0$ being the friction coefficient.

Outline

Outline:

(i) Implicit Programming Approach (ImP)

(ii) Solution of the state problem

(iii) Computation of subgradients (Sensitivity analysis)

(iv) Test examples

Implicit Programming Approach

Assume that

- f is continuously differentiable;
- U_{ad} is closed, and
- $\mathcal{F}$ is sufficiently small.

Then S is single-valued and locally lipschitz and (1) amounts to the mathematical program

$$\begin{array}{ll} \text{minimize} & \Theta(\alpha)\left(:= f(\alpha, S(\alpha))\right) \\ \text{subject to} & \\ & \alpha \in U_{ad} \end{array} \qquad (3)$$

in variable α only.

Implicit Programming Approach

(3) can be solved by a bundle method of NDO provided Θ is weakly semismooth.

Further, for each $\bar{\alpha} \in U_{ad}$ we must be able

- to compute $\bar{y} = S(\bar{\alpha})$, i.e., to solve a 3D contact problem with Coulomb friction;
- to compute one arbitrary Clarke subgradient

$$\xi \in \bar{\partial}\Theta(\bar{\alpha}).$$

Solution of the state problem

Let us replace the multiplier λ in $\tilde{Q}$ by a fixed slip bound $g \in \mathbb{R}^p_+$. Then (2) amounts to the optimization problem

$$\text{minimize } \frac{1}{2}\langle u, A(\alpha)u\rangle - \langle \ell(\alpha), u\rangle + \sum_{i=1}^{p} \mathcal{F} g^i \| u_\tau^i \|$$

subject to (4)

$$-u_\nu \leq \alpha .$$

We introduce a new variable $\vartheta \in \mathbb{R}^{2p}$ and put

$$\mu = \begin{bmatrix} \vartheta \\ \lambda \end{bmatrix}.$$

Then a dual problem to (4) attains the form

$$\begin{aligned}&\text{minimize } \tfrac{1}{2}\langle \mu, F\mu\rangle - \langle h, \mu\rangle\\ &\text{subject to}\\ &\qquad \|\vartheta^i\| \le \mathcal{F} g^i, \quad \lambda \ge 0,\end{aligned} \tag{5}$$

where F and h can be expressed in terms of $A(\alpha)$, $\ell(\alpha)$ and α.

Method of successive approximations:

1° Initialize g_0, put $k := 0$;

2° Solve (5) with $g = g_k$ and compute thus $\mu_k = (\vartheta_k, \lambda_k)$;

3° Put $g_{k+1} = \lambda_k$, $k = k+1$ and go to 2°.

In most MPECs S is a PC^1 mapping, i.e., S is continuous and for all α

$$S(\alpha) \in \{S_1(\alpha), \ldots, S_q(\alpha)\},$$

where $S_1, \ldots, S_q$ are continuously differentiable. In such a case one has for the Clarke generalized Jacobian $\bar{\partial} S(\bar{\alpha})$ the inclusion

$$\bar{\partial} S(\bar{\alpha}) \subset \operatorname{conv}\{\nabla S_i(\bar{\alpha}) \mid i \in I(\bar{\alpha})\},$$

where

$$I(\bar{\alpha}) := \{i \in \{1, 2, \ldots, q\} \mid S(\bar{\alpha}) = S_i(\bar{\alpha})\}$$

is the index set of active pieces.

Computation of subgradients (sensitivity analysis)

Consequently, on the basis of the Clarke generalized Jacobian Chain Rule, one may put

$$\xi = \nabla_\alpha f(\bar{\alpha}, \bar{y}) + (\nabla S_i(\alpha))^T \nabla_y f(\bar{\alpha}, \bar{y}) \quad \text{for some } i \in I(\bar{\alpha}).$$

Unfortunately, in case of 3D contact problem with Coulomb friction we do not dispose with a reprezentation of S showing that it is a PC^1 mapping.

Computation of subgradients (sensitivity analysis)

One could attempt to compute a matrix from $\bar{\partial} S(\bar{\alpha})$ according to the definition as

$$\lim_{\alpha_i \to \bar{\alpha}} \{\nabla S(\alpha_i) \,|\, S \text{ is differentiable at } \alpha_i\},$$

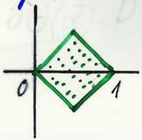
but it is not easy to identify all situations, when S is differentiable.

It holds that for all y^*

$$(\bar{\partial} S(\bar{\alpha}))^T y^* = \text{conv}\, D^*S(\bar{\alpha}, \bar{y})(y^*)$$

and so it suffices a vector from $D^*S(\bar{\alpha})(\nabla_y f(\bar{\alpha}, \bar{y}))$.

Example: $S(\cdot) = \mathrm{Proj}_{\mathcal{K}_2}(\cdot)$ at its vertex, $y^* = (1,0)^T$.



$\bar{\partial} S(0) y^*$

$D^* S(0)(y^*)$

Computation of $D^* S(\bar{\alpha})$:

(2) has the form

$$0 \in F(\alpha, y) + Q(y)$$

which yields

$$S(\alpha) = \left\{ y \,\middle|\, \begin{bmatrix} y \\ -F(\alpha, y) \end{bmatrix} \in \mathrm{gr}\, Q \right\}.$$

Computation of subgradients (sensitivity analysis)

Theorem 1. If $\mathcal{F}$ is sufficiently small, then (2) satisfies the strong regularity condition at all points $(\bar{\alpha}, \bar{y})$, where $\bar{\alpha} \in U_{ad}$ and $\bar{y} = S(\bar{\alpha})$. Consequently, for all y^*

$$D^*S(\bar{\alpha})(y^*) \subset \{(\nabla_\alpha F(\bar{\alpha}, \bar{y}))^T v \mid 0 \in y^* + (\nabla_y F(\bar{\alpha}, \bar{y}))^T v + D^*Q(\bar{y}, -F(\bar{\alpha}, \bar{y}))(v)\}.$$

Equality holds provided $\operatorname{gr} Q$ is normally reg. at $(\bar{y}, -F(\bar{\alpha}, \bar{y}))$.

For the purpose of computation of D^*Q we will reorder y as $(y^1, y^2, \ldots, y^p) \in (\mathbb{R}^4)^p$ with

$$y^i = (u_\tau^i, u_\nu^i, \lambda^i) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}_+, \quad i = 1, 2, \ldots, p$$

and reorganize accordingly the lines in (2). The multi-valued part attains then the form

$$Q(y) = \begin{bmatrix} \Phi(y^1) \\ \Phi(y^2) \\ \vdots \\ \Phi(y^p) \end{bmatrix} \quad \text{with } \Phi(y^i) = \begin{bmatrix} \mathcal{F}\lambda^i \partial \| u_\tau^i \| \\ 0 \\ N_{\mathbb{R}_+}(\lambda^i) \end{bmatrix}.$$

Computation of subgradients (sensitivity analysis)

Consequently, with $b = (b^1, b^2, \dots, b^p) \in Q(y)$ and $b^* = (b^{*1}, b^{*2}, \dots, b^{*p})$, $y^* = (y^{*1}, y^{*2}, \dots, y^{*p})$ one has

$$y^* \in D^*Q(y,b)(b^*) \iff y^{*i} \in D^*\Phi(y^i; b^i)(b^{*i})$$

for $i = 1, 2, \dots, p$.

So, it suffices to consider the ith node and investigate the mapping

$$\begin{bmatrix} b_1^i \\ b_2^i \end{bmatrix} \in \mathcal{F}\lambda^i \partial \| u_\tau^i \|$$

$$b_3^i = 0$$

$$b_4^i \in N_{\mathbb{R}_+}(\lambda^i).$$

We assume for simplicity that $\mathcal{F} = 1$, omit the index i and put $b_{12} = (b_1, b_2)^T$. The position of (y, b) in $\operatorname{Gr} \Phi$ is specified in the following table:

	strong contact $\lambda > 0,\ b_4 = 0$	weak contact $\lambda = 0,\ b_4 = 0$	no contact $\lambda = 0,\ b_4 < 0$
sliding $u_\tau \neq 0$	M_1	M_2	L
weak sticking $u_\tau = 0$ $\lVert b_{12} \rVert = \lambda$	M_3^-	M_4	L
strong sticking $u_\tau = 0$ $\lVert b_{12} \rVert < \lambda$	M_3^+		

Simple cases (M_1, L, M_3^+):

(M_1): $\exists$ a neighborhood $\mathcal{U}$ of the reference point $(\bar{u}_\tau, \bar{u}_\nu, \bar{\lambda}, \bar{b}_1, \bar{b}_2, 0, \bar{b}_4) \in \operatorname{gr} \Phi$ such that

$$u_\tau \neq 0, \ \lambda > 0, \ b_{12} = \frac{u_\tau}{\|u_\tau\|}, \ b_3 = 0 \ \text{on} \ \operatorname{gr} \Phi \cap \mathcal{U}.$$

Consequently,

$$\left(\nabla \Phi\left(\bar{u}_\tau, \bar{u}_\nu, \bar{\lambda}, \bar{b}\right)\right)^* = \begin{bmatrix} \bar{\lambda} \dfrac{(\bar{u}_{\tau 2})^2}{\|\bar{u}_\tau\|^3} & -\bar{\lambda} \dfrac{\bar{u}_{\tau 1} \bar{u}_{\tau 2}}{\|\bar{u}_\tau\|^3} & 0 & 0 \\ -\bar{\lambda} \dfrac{\bar{u}_{\tau 1} \bar{u}_{\tau 2}}{\|\bar{u}_\tau\|^3} & \bar{\lambda} \dfrac{(\bar{u}_{\tau 1})^2}{\|\bar{u}_\tau\|^3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \dfrac{\bar{u}_{\tau 1}}{\|\bar{u}_\tau\|} & \dfrac{\bar{u}_{\tau 2}}{\|\bar{u}_\tau\|} & 0 & 0 \end{bmatrix}.$$

L $\exists$ a neighborhood $\mathcal{U}$ of the reference point $(\bar{u}_\tau, \bar{u}_\nu, \bar{\lambda}, \bar{b}_1, \bar{b}_2, 0, \bar{b}_4) \in \operatorname{gr}\Phi$ such that

$$\lambda = 0, \qquad b_{12} = 0, \ b_4 < 0 \ \text{ on } \operatorname{gr}\Phi \cap \mathcal{U}.$$

It follows that

$$N_{\operatorname{gr}\Phi}(\bar{u}_\tau, \bar{u}_\nu, \bar{\lambda}, \bar{b}) = \{(\xi, \eta) \in \mathbb{R}^4 \times \mathbb{R}^4 \mid \xi_1 = \xi_2 = \xi_3 = 0, \ \eta_4 = 0\}.$$

Hence,

$$D^*\Phi(\bar{u}_\tau, \bar{u}_\nu, \bar{\lambda}, \bar{b})(\eta) = \begin{cases} \{0\} \times \{0\} \times \{0\} \times \mathbb{R} & \text{if } \eta_4 = 0 \\ \emptyset & \text{otherwise.} \end{cases}$$

Computation of subgradients (sensitivity analysis)

(M_3^+) $\exists$ a neighborhood $\mathcal{U}$ of the reference point such that

$$u_\tau = 0, \ \lambda > 0, \ \|b_{12}\| < \lambda, \quad b_4 = 0 \text{ on } \operatorname{gr}\Phi \cap \mathcal{U}.$$

It follows that

$$N_{\operatorname{gr}\Phi}(\bar{u}_\tau, \bar{u}_\nu, \bar{\lambda}, \bar{b}) = \{(\xi, \eta) \in \mathbb{R}^4 \times \mathbb{R}^4 \mid \xi_3 = \xi_4 = 0, \ \eta_1 = \eta_2 = 0\}.$$

Hence,

$$D^*\Phi(\bar{u}_\tau, \bar{u}_\nu, \bar{\lambda}, \bar{b})(\eta) = \begin{cases} \mathbb{R} \times \mathbb{R} \times \{0\} \times \{0\} & \text{if } \eta_1 = \eta_2 = 0 \\ \emptyset & \text{othewise.} \end{cases}$$

M_2 (weak contact, sliding)

Proposition 2. Let

$$F(x,y,z) = \begin{bmatrix} G(x,y) \\ H(y,z) \end{bmatrix},$$

where G, H are closed-graph multifunctions. Assume that

$$\begin{bmatrix} \bar{d}_1 \\ \bar{d}_2 \end{bmatrix} \in F(\bar{x},\bar{y},\bar{z}),$$

and the qualification condition

$$\begin{bmatrix} 0 \\ w_2 \end{bmatrix} \in D^*G(\bar{x},\bar{y})(0) \;\&\; \begin{bmatrix} -w_2 \\ 0 \end{bmatrix} \in D^*H(\bar{y},\bar{z})(0) \Rightarrow w_2 = 0$$

is fulfilled.

Then one has

$$D^*F(\bar{x},\bar{y},\bar{z})(d_1^*,d_2^*) \subset \{(w_1, w_2+w_3, w_4) \mid (w_1,w_2)\in D^*G(\bar{x},\bar{y})(d_1^*), (w_3,w_4)\in D^*H(\bar{y},\bar{z})(d_2^*)\}.$$

If G is continuously differentiable, then

$$D^*F(\bar{x},\bar{y},\bar{z})(d_1^*,d_2^*) =$$

$$\{((\nabla_x G(\bar{x},\bar{y}))^T d_1^*, (\nabla_y G(\bar{x},\bar{y}))^T d_1^* + w_3, w_4) \mid (w_3,w_4)\in D^*H(\bar{y},\bar{z})(d_2^*)\}.$$

Computation of subgradients (sensitivity analysis)

Proposition 3. Let the reference point $(\bar{y}, \bar{b})$, $\bar{y} = (\bar{u}_\tau, \bar{u}_\nu, \bar{\lambda})$ belong to M_2. Then one has

$$D^*\Phi(\bar{y}, \bar{b})(\eta) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{\bar{y}_1}{\|\bar{y}_{12}\|}\eta_1 + \dfrac{\bar{y}_2}{\|\bar{y}_{12}\|}\eta_2 + w \end{bmatrix} : \right.$$

$$\left. w \in \begin{cases} \mathbb{R} & \text{provided } \eta_4 = 0 \\ \mathbb{R}_- & \text{provided } \eta_4 < 0 \\ 0 & \text{otherwise.} \end{cases} \right\}.$$

approaching via L

M_2

M_1

Computation of subgradients (sensitivity analysis)

Remark. By approaching $(\bar{y}, \bar{b})$ via L we delete the contact condition which has no influence. This leads to the simplest adjoint GE and therefore this option has been used in the computer code. By approaching $(\bar{y}, \bar{b})$ via M_1 the contact condition amounts to an equality.

Computation of subgradients (sensitivity analysis)

M_3^- (strong contact, weak sticking)

Proposition 4. Consider a point $(\bar y, \bar b)$ from M_3^- and put $w = \frac{\bar b_{12}}{\lambda}$ $(\in S_2)$. Then one has

$$N_{\operatorname{gr}\Phi}(\bar y, \bar b) = \{(\xi, \eta) \in \mathbb{R}^4 \times \mathbb{R}^4 \mid \xi_{12} = 0,\ \xi_3 = 0,\ (\eta_{12}, \xi_4) \in \mathbb{R}(w, -1)\}$$

M_1 $$\cup \{(\xi, \eta) \in \mathbb{R}^4 \times \mathbb{R}^4 \mid \xi_{12} \in w^\perp,\ \xi_3 = 0,\ \xi_4 = 0,\ \eta_{12} = 0\}$$

M_3^+ $$\cup \{(\xi, \eta) \in \mathbb{R}^4 \times \mathbb{R}^4 \mid \xi_3 = 0,\ \xi_4 = 0,\ \eta_{12} = 0\}$$

M_3^- $$\cup \{(\xi, \eta) \in \mathbb{R}^4 \times \mathbb{R}^4 \mid \langle \xi_{12}, w \rangle \le 0,\ \xi_3 = 0,\ (\eta_{12}, \xi_4) \in \mathbb{R}_+(w, -1)\}.$$

Remark. The first two sets exhaust accumulation points of sequences $\xi^{(i)}, \eta^{(i)}$ such that

$$\xi^{(i)} = (\nabla \Phi(u_\tau^{(i)}, u_\nu^{(i)}, \lambda^{(i)}))^T \eta^{(i)},$$

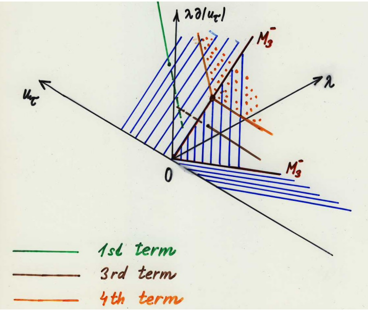
provided $u_\tau^{(i)} \to 0$ and $\lambda^{(i)} \to \bar{\lambda} > 0$.

Remark. By approaching $(\bar{y}, \bar{b})$ via M_3^+ we do not have to do with w and get a simple formula. Hence, this option is used in the computer code.

Computation of subgradients (sensitivity analysis)

Illustration of the above result in 2D, where

$$\Phi(u_\tau, u_\nu, \lambda) = \begin{bmatrix} \lambda \partial |u_\tau| \\ 0 \\ N_{\mathbb{R}_+}(\lambda) \end{bmatrix}$$



The 2nd term does not have an analogue in the 2D (polyhedral) case.

Test examples

$$f(\alpha, y) = \|\lambda - \bar{\lambda}_\nu\|^2,$$

where $\bar{\lambda}_\nu$ is a "desired" vector of normal stresses on the contact boundary.

The control vector α represents control points of Bezier surface.

U_{ad} comprises the constraint

$$vol(\Omega(\alpha)) = const.$$

and constraints restricting the slopes of Γ_c.

Test examples

Discretization : $25 \times 12 \times 12$ (3600 nodes)

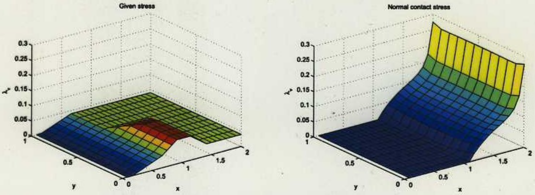
$p = 300$ (1200 state variables)

$\dim \alpha = 32$

stopping criterion : $\varepsilon = 10^{-4}$

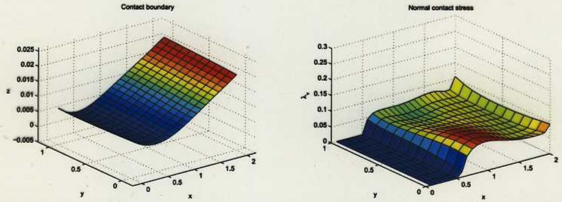
Test examples

Example: $\min \|\overline{\lambda}_\nu - \lambda_\nu\|_2^2$, $\varepsilon = 1\cdot 10^{-4}$

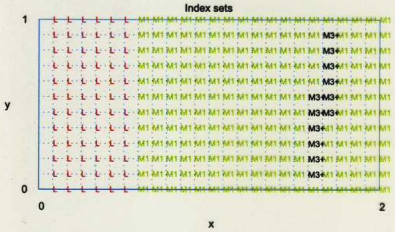


Target normal stress (left) and normal stress for initial design (right).

Test examples



Optimal design (left) and normal stress for optimal design (right).



Index sets for optimal design.

Open questions and further plans

Open questions

1) Semismoothness of S;
2) $D^*\Phi$ in the case of M_4 exactly.

Further plans

1) To replace BT by a faster solver;
2) To rewrite all parts of the code to one language.