A Note on Non-symmetric Independence Models

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Abstract

Some independence models not necessarily closed with respect to symmetry property are briefly recalled and they arise in different framework. The L-separation criterion for directed acyclic graphs is useful for effective description of such models. Since independence structures are richer than the graphical ones, the notion of minimal I-map has been redefined in this context and its properties are detected.

1 Introduction

In probability and statistics graphs are used to describe conditional independence structures (or analogously dependence/association structures) and to "simplify" the computations for learning structures and numerical evaluations. Actually, there are three main classic approaches based on *undirected graphs* [18], *directed acyclic graphs* [20], or *chain graphs* [22].

These graphical structures obey graphoid properties (i.e. symmetry, decomposition, weak union, contraction, intersection). However, in literature more general models have been introduced, which do not obey the above usual graphoid properties. In particular we focus on those not necessarily closed with respect to symmetric property (called briefly non-symmetric independence models). This means that a model $\mathcal{M}$ may contain an independence statement $X \perp\!\!\!\perp Y|Z$, but not the statement $Y \perp\!\!\!\perp X|Z$.

Thus, in the following we provide a very brief review of some non-symmetric independence models, which arise from different formalisms or uncertainty measures.

We stress as done in [21, 1] that the ordinary independence notion of probability theory is a concept without any "direction" and indicates usually a mutual association between two random variables or processes. We recall that independence models induced by a positive probability satisfy all the graphoid properties [10], while if the positive condition fails also the intersection property can fail.

This is connected with the fact that the classic definition of stochastic independence presents counter-intuitive situations when zero or one probability events are involved (see, for example, [6, 8]): for example, a possible event with zero or one probability is independent of itself. Zero probability values and logical constraints among random variables (which can be more general than linear dependence) are interesting not only from a merely theoretical point of view, but they are met in many real problems, for example in medical diagnosis [11], statistical mechanics, physics, etc. [17]. The counter-intuitive situations cannot be avoided within the usual framework of conditional probability, while in the more general framework (de Finetti [12], Dubins [14]), a definition of stochastic independence, which avoids these critical situations, has been introduced in [6] and the main properties connected with graphoid structures were proved in [23]: these independence models generally are not closed with respect to the symmetry property and this is due essentially to the presence of "unexpected events" (it means with zero probability).

Concerning other independence models arising in different uncertainty formalisms (as in possibility theory) we referee to the papers [3, 15, 4, 9, 26], where also in these cases some non-symmetrical structures come out.

Moreover, independence relations not necessarily symmetric have been introduced starting (as far as we know) from 70th years: we recall that ones related to some multistate stochastic processes (see e.g. [21, 1, 13]) applied to event history analysis and time series analysis. The crucial concept is that of local independence (introduced in [21] and generalized in [1]) and it allows a one-side influence (or, dually, independence), then it radically differs from the usual stochastic independence notion. For example, in a medical application the question is whether a particular disease of skin is related to the time of menopause and the analysis consists into claiming that menopause increases the risk of disease whereas the skin disease does not influence the menopause time. Let $X = \{X_t\}$ and $Y = \{Y_t\}$ be two stochastic processes and in [21] X and Y are time-continuos Markov chain on some finite state space where the two components cannot change value at the same time. The process Y is said locally independent of X over some time interval if the transition intensities for changes in Y are independent of the value of X for all time in the interval. Naturally, Y is locally dependent on X if it is not locally independent. In [1] the notion is extended to more general processes and it is compared with ortogonality concept.

Furthermore, in the context of special cases of event history analysis (as survival analysis) also logical constraints come out and this carries again to non-symmetric independence models, in fact the occurrence of a specific event marks the transition into an absorbing state: let $Y_t = 1$ describe the survival status and the covariate processes $X_1, ..., X_K$ the occurrence of intermediate events (as e.g. a side effect of a medicamentation) and we could have [2] that X_j ($j \in \{1, ..., K\}$) is not locally independent of Y since after death no further transitions are possible (so a logical constraints is present), while some covariates X_j are locally independent on the survival.

In this note, we show how to represent such non-symmetrical independence models (together with logical constraints) through directed acyclic graphs by using L-separation criterion [24]. Since these structures are richer than the graphical ones (see [25]): some independence models cannot be completely described by a graph. Hence, we provide in this context the notion of minimal I-map (a concept already well-known inside classic graphical models [20, 16]) and we show how to build it, by underling the main differences arising from the lack of symmetry property such minimal I-maps, and, in addition, we prove that any ordering on the variables gives rise to an I-map for any independence model $\mathcal{M}$ obeying to non necessarily symmetric graphoid properties. On the other hand, the ordering has a crucial role: in fact, if a perfect I-map (able to describe all the independence statements) exists, it can be built using only some specific ordering on the variables.

2 Basic graphical concepts

A *l-graph* G is a triplet $(V, E, \mathcal{B})$, where V is a finite set of *vertices*, E is a set of *edges* (i.e. a subset of ordered pairs of distinct vertices of $V \times V \setminus \{(v,v) \ : v \in V\}$) and $\mathcal{B}$ is a family (possibly empty) of subsets of vertices. The elements of the family $\mathcal{B} = \{B\ ,\ \ B \subseteq V\}$ are represented graphically by boxes enclosing the vertices in B. If $\mathcal{B}$ is empty, then the l-graph is a graph.

The attention in the sequel will be focused on *directed acyclic* l-graphs, and to introduce this kind of l-graphs we need to recall some basic notion from graph theory [18, 20].

A directed l-graph is a l-graph whose set of vertices E satisfies the following property: $(u,v) \in E \Rightarrow (v,u) \notin E$. A directed edge $(u,v) \in E$ is represented by an arrow pointing from u to v, $u \to v$. We say that u is a *parent* of v and v a *child* of u. The set of parents of v is denoted by $pa(v)$ and the set of children of u by $ch(u)$.

A *path* from u to v is a sequence of distinct vertices $u = u_1, \ldots, u_n = v$, $n \geq 1$ such that either $u_i \to u_{i+1}$ or $u_{i+1} \to u_i$ for $i = 1, \ldots, n-1$. A *directed path* from u to v is a sequence $u = u_1, \ldots, u_n = v$ of distinct vertices such that $u_i \to u_{i+1}$ for all $i = 1, \ldots, n-1$. If there is a directed path from u to v, we say that u is an ancestor of v or v a descendant of u and we write $u \mapsto v$. The symbols $an(v)$ and $ds(u)$ denote the set of *ancestors* of v and the set of *descendants* of u (vertices that $u \in an(v)$ and $v \in ds(u)$), respectively. Note that, according to our definition, a sequence consisting of one vertex is a directed path of length 0, and therefore every vertex is its own descendent and ancestor, i.e. $u \in an(u), u \in ds(u)$.

A *reverse directed path* from u to v is a sequence $u = u_1, \ldots, u_n = v$ of distinct vertices such that $u_i \leftarrow u_{i+1}$ for all $i = 1, \ldots, n-1$.

A *n-cycle* is a sequence of $u_1, \ldots, u_n$, with $n > 3$, such that $u_n \to u_1$ and $u_1, \ldots, u_n$ is a directed path. A directed graph is *acyclic* if it contains no cycles.

Given an acyclic directed graph G, the relation $\mapsto$ defines a *partial ordering* $\prec_G$ on the set of vertices, in particular for any $u, v \in V$ we have that if $u \in an(v)$, then $u \prec_G v$, while if $u \in ds(v)$, then $v \prec_G u$.

2.1 L-graphs and logical constraints

To visualize which variables are linked by a logical constraint we need to refer to the family $\mathcal{B}$ of subsets of vertices. Since, given a random vector $X = (X_1, \ldots, X_n)$, a vertex i is associated with each random variable X_i, by means of the boxes $B \in \mathcal{B}$, we visualize the sets of random variables linked by a logical constraint (more precisely, a logical constraint involves the events of the partitions generated by the random variables). Recall that the partitions $\mathcal{E}_1, \ldots, \mathcal{E}_n$ are *logically independent* if for every choice $C_i \in \mathcal{E}_i$, with $i = 1, ..., n$, the conjunction $C_1 \wedge \ldots \wedge C_n \neq \emptyset$.

Obviously, if n partitions are logically independent, then arbitrary subsets of these partitions are logically independent.

However, n partitions $\mathcal{E}_1, \ldots, \mathcal{E}_n$ need not be logically independent, even if every $n-1$ partitions can be logically independent; it follows that there is a *logical constraint* such that an event of the kind $C_1 \wedge \ldots \wedge C_n$ is impossible, with $C_i \in \mathcal{E}_i$. For example, suppose $\mathcal{E}_1 = \{A, A^c\}$, $\mathcal{E}_2 = \{B, B^c\}$ and $\mathcal{E}_3 = \{C, C^c\}$ are three distinct partitions of Ω with $A \wedge B \wedge C = \emptyset$. All the couples of that partitions are logically independent, but they are not logically independent. Actually, the partition $\mathcal{E}_1$ is not logically independent of the partition generated by $\{\mathcal{E}_2, \mathcal{E}_3\}$. The same conclusion is reached replacing $\mathcal{E}_1$ by $\mathcal{E}_2$ or $\mathcal{E}_3$.

Given n partitions and some logical constraints among such partitions, it is possible, for each constraint, to find the *minimal subset* $\{\mathcal{E}_1, \ldots, \mathcal{E}_k\}$ of partitions generating it. Actually, $\mathcal{E}_1, \ldots, \mathcal{E}_k$ are such that $C_1 \wedge \ldots \wedge C_k = \emptyset$, with $C_i \in \mathcal{E}_i$, and, in addition, for all $j = 1, \ldots, k$, $C_1 \wedge \ldots \wedge C_{j-1} \wedge C_{j+1} \wedge \ldots \wedge C_k \neq \emptyset$. Such set of partitions $\{\mathcal{E}_1, \ldots, \mathcal{E}_k\}$ is said the *minimal set* generating the given logical constraint, and it is singled-out graphically by the box $B = \{1, \ldots, k\}$, which includes exactly the vertices associated to the corresponding random variables $X_1, \ldots, X_k$. Then, in the sequel we call the boxes $B \in \mathcal{B}$ *logical components*.

2.2 Separation criterion for directed acyclic graphs

To represent non-symmetric independence models we need to recall L-separation criterion . In fact, the classic separation criterion for directed acyclic graphs (see [20]), known as d-separation (where d stands for directional), is not suitable for our purposes, because it induces a graphoid structure, and so it is not useful to describe a model where symmetry property may not hold.

Definition 1 *Let G be an acyclic directed graph. A path $u_1, \ldots, u_n$, $n \geq 1$ in G is blocked by a set of vertices $S \subset V$, whenever there exists $1 < i < n$ such that one of the following three condition holds:*

1. $u_{i+1} \to u_i \to u_{i-1}$ *(i.e. u_{i-1}, u_i, u_{i+1} is the reverse directed path) and* $u_i \in S$
2. $u_{i-1} \leftarrow u_i \to u_{i+1}$ *and* $u_i \in S$
3. $u_{i-1} \to u_i \leftarrow u_{i+1}$ *and* $ds(u_i) \notin S$

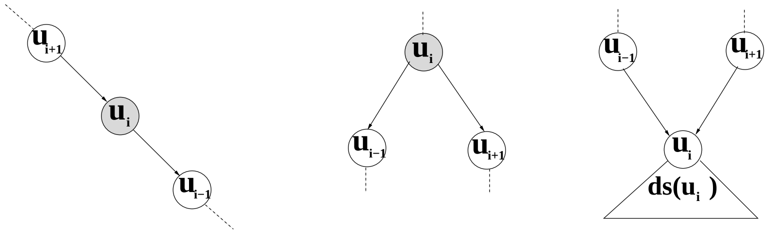

Figure 1: Blocked paths

The three conditions of Definition 1 are illustrated by Figure 1 (the grey vertices belong to S). Note that the definition of blocked path strictly depends on the direction of the path, in fact the main difference between our notion and that used in d-separation criterion [20] consists essentially in condition 1 of Definition 1. The path u_{i-1}, u_i, u_{i+1} drawn in the left-side of Figure 1 is blocked by u_i, while its reverse is not blocked by u_i because of the direction. Hence, the reverse path of a blocked one is not necessarily blocked according to our definition, so the blocking path notion does not satisfy the symmetry property.

The second and third cases of Definition 1 are like in d-separation criterion.

Definition 2 *Let G be a directed acyclic l-graph and let U, W and S be three pairwise disjoint sets of vertices of V. We say that U is L-separated from W by S in G and write symbol $(U, W|S)^l_G$, whenever every path in G from U to W is blocked by S and moreover, the following "logical separation" condition holds*

$$\forall B \in \mathcal{B} \text{ s.t. } B \subseteq U \cup W \cup S \text{ one has either } B \cap U = \emptyset \text{ or } B \cap W = \emptyset. \quad (1)$$

Figure 2 clarifies when condition (1) holds (the set of vertices V_i and S are represented as ovals).

Figure 2: Representation of logical components: in the left-side V_1 and V_2 are not connected, in the right-side they are connected by B

Since the notion of blocked path is not necessarily symmetric, it follows that $(U, W|S)^l_G \not\Rightarrow (W, U|S)^l_G$. Actually, the lack of symmetry property depends on the notion of blocked path and not on the condition of logical separation (1).

Theorem 1 *[24] Let $G = (V, E, \mathcal{B})$ be a graph. The following properties hold*

1. *(Decomposition property)*

$$(U, W \cup Z|S)^l_G \Longrightarrow (U, W|S)^l_G$$

2. *(Reverse decomposition property)*

$$(U \cup Z, W|S)^l_G \Longrightarrow (U, W|S)^l_G$$

3. *(Weak union property)*

$$(U, W \cup Z|S)^l_G \Longrightarrow (U, W|Z \cup S)^l_G$$

4. *(Reverse weak union property)*

$$(U \cup Z, W|S)^l_G \Longrightarrow (U, W|Z \cup S)^l_G.$$

5. *(Contraction property)*

$$(U, W|S)^l_G \ \& \ (U, Z|W \cup S)^l_G \Longrightarrow (U, W \cup Z|S)^l_G$$

6. *(Reverse contraction property)*

$$(U, W|S)^l_G \ \& \ (Z, W|U \cup S)^l_G \Longrightarrow (U \cup Z, W|S)^l_G$$

7. *(Intersection property)*

$$(U, W|Z \cup S)^l_G \ \& \ (U, Z|W \cup S)^l_G \Longrightarrow (U, W \cup Z|S)^l_G$$

8. *(Reverse intersection property)*

$$(U, W|Z \cup S)^l_G \ \& \ (Z, W|U \cup S)^l_G \Longrightarrow (U \cup Z, W|S)^l_G$$

3 Minimal I-map

Given an independence model $\mathcal{M}$ over a set of variables (possibly) linked by a set of logical constraints, we look for a directed acyclic l-graph G describing all the statements T in $\mathcal{M}$ and localizing the set of variables involved in some logical constraint. But, generally, it is not always feasible to have such graph G (i.e. describing all the independence statements) for a given $\mathcal{M}$ as it happens for the classical independence models and also for non-symmetrical ones, see [23] for independence model in a coherent setting: as well as the independence model $\mathcal{M}_P$ containing the statements $(X_3, X_4) \perp\!\!\!\perp X_1|X_2$, $X_3 \perp\!\!\!\perp X_4|X_2$, $X_3 \perp\!\!\!\perp X_4|(X_1, X_2)$; $X_4 \perp\!\!\!\perp X_3|X_2$, $X_4 \perp\!\!\!\perp X_3|(X_1, X_2)$, $X_3 \perp\!\!\!\perp X_4$, $X_4 \perp\!\!\!\perp X_3$.

Note that $\mathcal{M}_P$ is not completely representable by a directed acyclic l-graph.

Hence, analogously as in [20], the notion of I-map is needed to be introduced.

Definition 3 *A directed acyclic l-graph G is an I-map for a given independence model $\mathcal{M}$ iff every independence statement represented by means of L-separation criterion in G is also in $\mathcal{M}$.*

Thus an I-map G for $\mathcal{M}$ may not represent every statement of $\mathcal{M}$, but those represented belong actually to $\mathcal{M}$, it means that the set $\mathcal{M}_G$ of statements described by G is contained in $\mathcal{M}$.

An I-map G for $\mathcal{M}$ is said *minimal* if removing any arrow from the l-graph G the obtained l-graph will no longer be an I-map for $\mathcal{M}$.

Given an independence model $\mathcal{M}$ over a random vector $(X_1, ..., X_n)$, let $\pi = (\pi_1, ..., \pi_n)$ be any ordering of the given variables, and, in addition, for any j, let $U_{\pi_j} = \{\pi_1, ..., \pi_{j-1}\}$ be the set of indexes before π_j, and D_{π_j} the minimal subset of U_{π_j} such that $X_{\pi_j} \perp\!\!\!\perp X_{R_{\pi_j}} | X_{D_{\pi_j}}$ where $R_{\pi_j} = U_{\pi_j} \setminus D_{\pi_j}$. Moreover, let W_{π_j} the minimal subset of U_{π_j} such that $X_{S_{\pi_j}} \perp\!\!\!\perp X_{\pi_j} | X_{W_{\pi_j}}$ where $S_{\pi_j} = U_{\pi_j} \setminus W_{\pi_j}$.

The subset $\Theta_\pi = \{X_{\pi_j} \perp\!\!\!\perp X_{R_{\pi_j}} | X_{D_{\pi_j}}\ ,\ X_{S_{\pi_j}} \perp\!\!\!\perp X_{\pi_j} | X_{W_{\pi_j}}\ :\ j = 1, ...n\}$ is said the *basic list* of $\mathcal{M}$ relative to π. From the basic list Θ_π and the set of logical components $\mathcal{B}$, a directed acyclic l-graph G (related to π) is obtained by drawing the boxes $B \in \mathcal{B}$ and designating D_{π_j} as parents of vertex π_j (for any vertex $v \in D_{\pi_j}$, an arrow goes from v to π_j), moreover, for any vertex $\pi_i \in W_{\pi_j} \setminus D_{\pi_j}$ check if the are only directed paths from π_i to π_j otherwise draw an arrow from π_i to π_j.

This construction of G from the basic list differs from the classic construction given for directed acyclic graphs with d-separation [20] essentially for the second part, which is useful to avoid the introduction of symmetric statements not in the given independence model.

For example, consider the independence model $\mathcal{M} = \{X_1 \perp\!\!\!\perp X_3 | X_2\}$ and considering the ordering $\pi = (2, 3, 1)$, the related directed acyclic l-graph is obtained following these steps: draw an arrow from 2 to 3, then consider the vertex 1 and draw an arrow from 2 to 1; now since $D_3 = \{2\}$, but the statement $X_3 \perp\!\!\!\perp X_1 | X_2$ is not in $\mathcal{M}$, we must draw an arrow from 3 to 1.

Now, we must prove that such directed acyclic l-graph obtained from the basic list Θ_π is an I-map for $\mathcal{M}$.

Theorem 2 *Let $\mathcal{M}$ be an independence model over a set of random variables linked by a set of logical constraints. Given an ordering π on the random variables, if $\mathcal{M}$ is an a-graphoid (i.e. closed with respect to decomposition, weak union, contraction, intersection and their reverse properties), then the directed acyclic l-graph G generated by the basic list Θ_π is an I-map for $\mathcal{M}$.*

Proof: For an a-graphoid of one variable it is obvious that the directed acyclic l-graph is an I-map. Suppose for a-graphoid structure with less than k variables that the directed acyclic l-graph is an I-map.

Let $\mathcal{M}$ be an independence model under k variables. Given an ordering π on the variables, let X_n be the last variable according to π (n denotes the vertex in G associated to X_n), $\mathcal{M}'$ the a-graphoid formed by removing all the independence statements involving X_n from $\mathcal{M}$ and G' the directed acyclic l-graph formed by removing n and all the arrows going to n (they cannot depart from n because is the last vertex) in G.

Since X_n is the last variable in the ordering π, it cannot appear in any set of parents D_{π_j} and it cannot appear in any W_{π_j} (with $j < k$), and the basic list $\Theta' = \Theta \setminus \{X_n \perp\!\!\!\perp X_{R_n} | X_{D_n}, X_{S_n} \perp\!\!\!\perp X_n | X_{W_n}\}$ generates $\mathcal{G}'$. Since $\mathcal{M}'$ has $k-1$ variables, $\mathcal{G}'$ is an I-map of it.

G is an I-map of $\mathcal{M}$ iff the set $\mathcal{M}_G$ of the independence statements represented in G by L-separation criterion is also in $\mathcal{M}$.

If X_n does not appear in T, then, being $T = (X_I \perp\!\!\!\perp X_J | X_K) \in \mathcal{M}_G$, T must be represented also in $\mathcal{G}'$, if it were not, then there would be a path in $\mathcal{G}'$ from I to J that is not blocked (according to L-separation) by K. But then it must be not blocked also in G, since the addition of a vertex and some arrows going to the new vertex cannot block a path. Since $\mathcal{G}'$ is an I-map of $\mathcal{M}'$, T must be an element of it, but $\mathcal{M}' \subset \mathcal{M}$, so $T \in \mathcal{M}$.
Otherwise (if X_n appears in T), T falls into one of the following situations. First of all, suppose that $T = ((X_I, X_n) \perp\!\!\!\perp X_J | X_K) \in \mathcal{M}_G$, let $X_n \perp\!\!\!\perp X_{R_n} | X_{D_n} \in \mathcal{M}$ (by construction). Obviously J and D_n have no vertices in common, otherwise we would have a path from a vertex in $j \in J \cap D_n$ pointing to n, so by L-separation n would not be separated from J given K in G.
Since there is an arrow from every vertex in D_n to n and every path from n to J is blocked by K in G, then every path from D_n to J must be blocked by K in G. Therefore, every path from both D_n and I to J are blocked by K in G. Now, if there is a logical component $B \in \mathcal{B}$ such that $B \subseteq D_n \cup I \cup J \cup K$ and both $B \cap (D_n \cup I)$ and $B \cap J$ are not empty, then remove a suitable vertex in B from D_n, w.l.g. Hence, the statement $(X_I, X_{D_n}) \perp\!\!\!\perp X_J | X_K$ belongs to $\mathcal{M}_G$. This statement does not contain the variable X_n, hence, being G' an I-map for $\mathcal{M}' \subset \mathcal{M}$, then $(X_I, X_{D_n}) \perp\!\!\!\perp X_J | X_K \in \mathcal{M}$.
Since $\mathcal{M}$ is closed under a-graphoid properties, (by weak union property) $X_n \perp\!\!\!\perp X_J | (X_I, X_{D_n}, X_K) \in \mathcal{M}$ and it follows $(X_I, X_{D_n}, X_n) \perp\!\!\!\perp X_J | X_K \in \mathcal{M}$ (using reverse contraction property), so $(X_I, X_n) \perp\!\!\!\perp X_J | X_K \in \mathcal{M}$ by decomposition property.

Now, suppose that $T = (X_I \perp\!\!\!\perp (X_J, X_n) | X_K) \in \mathcal{M}_G$, it means, by definition of L-separation and from the assumption that n is the last vertex in the ordering, that every path going from I to $J \cup n$ is L-separated by K. Therefore, when $((X_J, X_n) X_I \perp\!\!\!\perp | X_K) \in \mathcal{M}_G$, the proof goes in the same line of that in previous step. Otherwise, there is at least a path as in condition 1 of Definition 5. Since there is a subset $W_n \subseteq U_n$ such that every path between n and $I \cup K$ is blocked by W_n. Note that, $W_n = W^1 \cup W^2$ (W^1 or W^2 can be empty) with $W^2 \subseteq D_n$. Hence, W^2 and I cannot have common vertices. Moreover, let $J = J^1 \cup J^2 \cup J^3$ (J^1 or J^2 or J^3 can be empty) with $T_2 = (X_I \perp\!\!\!\perp (X_{W_n}, X_{J^1}) | X_K) \in \mathcal{M}_G$, while $J^2 \subseteq W^2$ and $J^3 = J \setminus (J^1 \cup J^2)$. Therefore, by construction, one has that every path between $n \cup J^3$ and $I \cup K \cup J^1$ is blocked by W_n. Hence, one has that from the previous step $(X_I, X_K, X_{J^1}) \perp\!\!\!\perp (X_n, X_{J^3}) | X_{W_n}$ and its symmetric statement belong to $\mathcal{M}$. Therefore, one has $X_I \perp\!\!\!\perp (X_n, X_{J^3}) | (X_{W_n}, X_K, X_{J^1}) \in \mathcal{M}$ by weak union property. Since $T_2 \in \mathcal{M}_G$ does not involve n, $T_2 \in \mathcal{M}$, so the statement $X_I \perp\!\!\!\perp (X_n, X_{W_n}, X_{J_1}, X_{J_3}) | X_K)$ belong to $\mathcal{M}$ (by contraction property), and it follows that $X_I \perp\!\!\!\perp (X_n, X_J) | X_K$ belongs to $\mathcal{M}$ (by reverse decomposition).

Now, the last cases that can happen is $T = (X_I \perp\!\!\!\perp X_J | (X_K, X_n)) \in \mathcal{M}_G$. It

must be the case that I is L-separated by J given K in G for if it were not, then there would be a path from some vertex in I to some vertex in J not passing trough K. But I is separated by J given n and K, so this path would pass through n; but n is the last vertex in the ordering, so all arrows go on it. Hence, it cannot block any unblocked path, and so $T_1 = (X_I \perp\!\!\!\perp X_J | X_K) \in \mathcal{M}_G$.
The statements T_1 and T imply that either $(X_I, X_n) \perp\!\!\!\perp X_J | X_K$ or $X_I \perp\!\!\!\perp (X_J, X_n) | X_K$ holds in G: in fact, if both I and J are connected to n, since n is the last vertex (from n an arrow cannot leave), then there is a directed path from I to n and another from J to n, so that one would get $X_I \perp\!\!\!\perp X_J | (X_K, X_n) \notin \mathcal{M}_G$. So, the conclusion follows by the previous steps.

Example 1 – Consider the independence model $\mathcal{M}_P$ over the binary random variables X_i with $i = 1, ..., 4$, such that $(X_1 = 1) \subseteq (X_2 = 1)$, containing the following statements $(X_3, X_4) \perp\!\!\!\perp X_1 | X_2$, $X_3 \perp\!\!\!\perp X_4 | X_2$, $X_3 \perp\!\!\!\perp X_4 | (X_1, X_2)$; $X_4 \perp\!\!\!\perp X_3 | X_2$, $X_4 \perp\!\!\!\perp X_3 | (X_1, X_2)$, $X_3 \perp\!\!\!\perp X_4$, $X_4 \perp\!\!\!\perp X_3$. This independence model can arise from a conditional probability, see [25]. The following pictures show the minimal I-map obtained by means of the proposed procedure for two possible orderings: $(1, 2, 3, 4)$ on the left-side and $(3, 4, 1, 2)$ on the right-side

Figure 3: Two possible I-Maps for the independence model $\mathcal{M}_P$ of Example 1

Actually, the picture in the left-side represents the independence statements $(X_3, X_4) \perp\!\!\!\perp X_1 | X_2$, $X_3 \perp\!\!\!\perp X_4 | X_2$, $X_4 \perp\!\!\!\perp X_3 | X_2$ and those implied by a-graphoid properties; while that one on the right-side describes the statement $X_3 \perp\!\!\!\perp X_4$ and its symmetric one. Note that these two graphs actually are minimal I-maps; in fact removing any arrow from them, we may read independence statements not in $\mathcal{M}_P$. The block $B = \{1, 2\}$ localizes the logical constraint $A_1 \subset A_2$.

If for a given independence model over n variables there exists a perfect map G, then (at least) one of $n!$ orderings among the variables will generate the l-graph G. More precisely, such orderings, which give rise to G, are all the orderings compatible with the partial order induced by G.

4 Conclusions

The L-separation criterion for directed acyclic graphs has been recalled together with its main properties. This is very useful for effective description of independence models induced by different uncertainty measures [4, 5, 6, 7, 8, 19, 23, 25, 26]. In fact, these models cannot be represented efficiently by the well-known graphical models [18, 20], because the related separation criteria satisfy the symmetry property.

By considering the L-separation criterion introduced in [23], we show that for some independence models there is not a perfect map even using L-separation criterion. Therefore, the notion of minimal I-map has been redefined in this context and we have shown how to build it given an ordering on the variables. In addition, we have proved that for any ordering on the variables there is a minimal I-map for a given independence model obeying to asymmetric graphoid properties.

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