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Stochastic wave equation with critical nonlinearities: Temporal regularity and uniqueness ☆

Martin Ondreját [1]

Institute of Information Theory and Automation, Pod Vodárenskou věží 4, CZ-182 08, Praha 8, Czech Republic

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ABSTRACT

We consider a nonlinear wave equation $u_{tt} = \Delta u + f(u) + g(u)\dot{W}$ on $\mathbb{R}^d$ driven by a spatially homogeneous Wiener process W with a finite spectral measure and with nonlinear terms f, g of critical growth. We study pathwise uniqueness and norm continuity of paths of (u, u_t) in $H^1(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$ under the hypothesis that there exists a local solution u such that (u, u_t) has weakly continuous paths in $H^1(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$.

1. Introduction

Wave equations $u_{tt} = \Delta u + f(u) + g(u)\dot{W}$ subject to random excitations have been thoroughly studied in last years for its applications in physics, relativistic quantum mechanics and oceanography (see e.g. [1–3,5,7–9,12–17,19,20,22–24] and references therein). The randomness in these equations has been predominantly modeled by spatially homogeneous Wiener processes, i.e. by centered Gaussian processes $(W(t,x)\colon\ t \geqslant 0,\ x \in \mathbb{R}^d)$ satisfying

$$\mathbb{E}W(t,x)W(s,y) = (t \wedge s)\Gamma(x-y), \quad t,s \geqslant 0,\ x,y \in \mathbb{R}^d,$$

for some function or even a distribution Γ called *the spatial correlation* of W (see e.g. [25] for details).

Except for the works [5,16,19,20] and [22], the functions $f, g : \mathbb{R} \to \mathbb{R}$ were assumed to be globally Lipschitz in the above cited papers. Global lipschitzianity of the real functions f and g allows one to study the equation on the state space $L^2(\mathbb{R}^d) \oplus H^{-1}(\mathbb{R}^d)$ where the Nemytski operators associated to f and g are globally Lipschitz, the spatial correlation Γ may be very general, e.g. a distribution or at least a continuous function possibly unbounded at the origin, and, consequently, the solution

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E-mail address: ondrejat@utia.cas.cz.
[1] Fax: +420 286 890 378.

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and its derivative (u, u_t) take values in $L^2(\mathbb{R}^d) \oplus H^{-1}(\mathbb{R}^d)$. If $f, g : \mathbb{R} \to \mathbb{R}$ are only locally Lipschitz then various techniques of proof of existence of solutions – usually based on Lyapunov functions, energy estimates, Sobolev embeddings or the fine Strichartz inequality – require the state space for the solution to be the so-called "energy space" $H^1(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$ (see e.g. [26]), the spatial correlation Γ to be a bounded function and *the spectral measure* $\mu = (2\pi)^{\frac{d}{2}}\widehat{\Gamma}$ to be a finite measure (cf. equality (4.1) in this paper).

There is a few results concerning the stochastic wave equation

$$u_{tt} = \Delta u + f(u) + g(u)\dot{W}, \qquad u(0) = u_0, \qquad u_t(0) = v_0 \tag{1.1}$$

on $\mathbb{R}^d$ where $f, g : \mathbb{R} \to \mathbb{R}$ are general locally Lipschitz (or even merely continuous) functions that we will survey, for purposes of instructiveness and notational simplicity, just for the equation with polynomial nonlinearities

$$u_{tt} = \Delta u - u|u|^{p-1} + |u|^{\tilde{p}}\dot{W}, \qquad u(0) = u_0, \qquad u_t(0) = v_0. \tag{1.2}$$

It is known that global weak solutions (weak both in the probabilistic and in the PDE sense) exist provided that (u_0, v_0) is an $\mathscr{F}_0$-measurable $[H^1(\mathbb{R}^d) \cap L^{p+1}(\mathbb{R}^d)] \oplus L^2(\mathbb{R}^d)$-valued random variable, W is a spatially homogeneous Wiener process with bounded spectral correlation Γ (i.e. $\mu = (2\pi)^{\frac{d}{2}}\widehat{\Gamma}$ must be a finite measure) and

$$1 \leqslant \tilde{p} \leqslant \frac{p+1}{2} < \infty. \tag{1.3}$$

In this case, paths of (u, u_t) are known to take values in $[H^1(\mathbb{R}^d) \cap L^{p+1}(\mathbb{R}^d)] \oplus L^2(\mathbb{R}^d)$ and to be weakly continuous in $H^1(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$. Pathwise uniqueness is in general an open problem. These results were proved in [19] in a more general setting under, however, more stringent assumption $1 \leqslant \tilde{p} < \frac{p+1}{2} < \infty$. An extension to the critical case $\tilde{p} = \frac{p+1}{2}$ is possible (details will be given in a separate note).

Better results are known provided that either $d \leqslant 2$, or $d \geqslant 3$ and $p, \tilde{p} \in [1, \frac{d}{d-2}]$. If (u_0, v_0) is an $\mathscr{F}_0$-measurable $H^1(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$-valued random variable and W is a spatially homogeneous Wiener process with bounded spectral correlation Γ then there exists a local mild $H^1(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$-valued solution of (1.2), any weak $H^1(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$-valued local solution of (1.2) in the PDE sense is a mild $H^1(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$-norm continuous local solution, pathwise uniqueness holds in the class of $H^1(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$-valued solutions and the unique solution is global provided that (1.3) holds (see [20]).

Since the case $d \leqslant 2$ was resolved in the above cited works satisfactorily, the attention was paid to the case $d \geqslant 3$. It was proved that pathwise uniqueness holds among weak (in the PDE sense) local solutions of (1.2) with $H^1(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$-weakly continuous paths if $p < \frac{d+2}{d-2}$ and $\tilde{p} < \frac{d+1}{d-2}$ (i.e. the subcritical case) provided that the spectral measure $\mu = (2\pi)^{\frac{d}{2}}\widehat{\Gamma}$ of the spatially homogeneous Wiener process W is finite and has a finite εth-moment for some small positive $\varepsilon = \varepsilon(d, p, \tilde{p})$ (see [22]).

The aim of the present paper is to deal with the remaining problems in the subcritical case and problems in the critical case $p \leqslant \frac{d+2}{d-2}$ and $\tilde{p} \leqslant \frac{d+1}{d-2}$ (we remark that the critical case has not been studied yet at all). Thus, we study

(1) pathwise uniqueness in the class of adapted processes with $H^1(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$-weakly continuous paths,
(2) conditions under which local solutions with $H^1(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$-weakly continuous paths have $H^1(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$-norm continuous paths.

It shows up that both (1) and (2) hold in the class of adapted processes $z=(u,v)$ with $H^1(\mathbb{R}^d)\oplus L^2(\mathbb{R}^d)$-weakly continuous paths such that

$$u\in L^q_{\mathrm{loc}}\big(\mathbb{R}_+;\dot B^{\frac12}_q\big(\mathbb{R}^d\big)\big) \tag{1.4}$$

where $q=2\frac{d+1}{d-1}$ and $\dot B^{\frac12}_q(\mathbb{R}^d)$ is the homogeneous Besov space, provided that the spectral measure $\mu=(2\pi)^{\frac d2}\widehat{\Gamma}$ of the spatially homogeneous Wiener process W is finite and has a finite εth-moment for some small positive $\varepsilon=\varepsilon(d,p,\tilde p)$.

This result makes the subcritical case $p<\frac{d+2}{d-2}$, $\tilde p<\frac{d+1}{d-2}$ completely resolved since the additional hypothesis (1.4) is always satisfied in this case (see [22]). Yet, the critical case $p=\frac{d+2}{d-2}$, $\tilde p=\frac{d+1}{d-2}$ is resolved by this result only partially as the problem of existence of a solution of (1.2) satisfying (1.4) is still open.

Let us also present the state of art for the deterministic Cauchy problem

$$u_{tt}=\Delta u-u|u|^{p-1},\qquad u(0)=u_0,\qquad u_t(0)=v_0 \tag{1.5}$$

on $\mathbb{R}^d$ for $d\geqslant 3$. It is known that there exists a global solution u of (1.5) such that the path of (u,u_t) is weakly continuous in $H^1(\mathbb{R}^d)\oplus L^2(\mathbb{R}^d)$ if $p\in[1,\infty)$, $u_0\in H^1(\mathbb{R}^d)\cap L^{p+1}(\mathbb{R}^d)$ and $v_0\in L^2(\mathbb{R}^d)$ (see [27] or [29]), and this solution is norm continuous in $H^1(\mathbb{R}^d)\oplus L^2(\mathbb{R}^d)$ and unique in the class $C_w(\mathbb{R}_+;H^1(\mathbb{R}^d)\oplus L^2(\mathbb{R}^d))$ provided that $p<\frac{d+2}{d-2}$ (see [11]). If $p=\frac{d+2}{d-2}$ then there exists a unique global solution of (1.5) in the class

$$\big\{u\colon (u,u_t)\in C_w\big(\mathbb{R}_+;H^1\big(\mathbb{R}^d\big)\oplus L^2\big(\mathbb{R}^d\big)\big),\ u\in L^q_{\mathrm{loc}}\big(\mathbb{R}_+;\dot B^{\frac12}_q\big)\big\}$$

where $q=2\frac{d+1}{d-1}$ provided that $u_0\in H^1(\mathbb{R}^d)$ and $v_0\in L^2(\mathbb{R}^d)$ (see [28]).

We aim at proving the stochastic equivalent of the above mentioned PDE results. We, however, remark at this point that proofs of $H^1(\mathbb{R}^d)\oplus L^2(\mathbb{R}^d)$-norm continuity of solutions of deterministic wave equations with subcritically and critically growing nonlinearities are based on time reversal (see [11] and [28]) and this approach is not transferable to the case of SPDEs because of the presence of a nondecreasing filtration on a probability space. Our proof is therefore conducted in a different way.

We also remark, for completeness, that neither uniqueness nor temporal regularity is known to hold for (1.5) if $p>\frac{d+2}{d-2}$ or for (1.1) if $p>\frac{d+2}{d-2}$ or $\tilde p>\frac{d+1}{d-2}$, and the energy functional (7.4) is not well defined on the whole energy space $H^1\oplus L^2$ either. These are just few reasons why the exponents $p=\frac{d+2}{d-2}$ and $\tilde p=\frac{d+1}{d-2}$ are called critical.

Results of this paper are based on the Strichartz inequalities for the wave group (T_t) and on estimations of the γ-radonifying norm of $T_t\circ M_h$ where $M_h:\xi\to h\cdot\xi$ is a multiplication operator from the RKHS of the Wiener process to a homogeneous Besov or a Lebesgue space. These (and other preliminary and technical) results are collected in Appendix A of this paper.

2. Notation and conventions

The following notation and conventions will be used throughout this paper.

- $\mathscr{S}_{\mathbb{R}}$, $\mathscr{S}_{\mathbb{C}}$ denote the separable Fréchet spaces of rapidly decreasing real/complex functions on $\mathbb{R}^d$ equipped with the pseudonorms

$$\|\varphi\|_m=\sup_{x\in\mathbb{R}^d,\,|\alpha|\leqslant m}\big(1+|x|^m\big)\big|D^\alpha\varphi(x)\big|,\quad m\geqslant 1,$$

- $\mathscr{S}'_{\mathbb{R}}$, $\mathscr{S}'_{\mathbb{C}}$ denote the topological dual spaces to $\mathscr{S}_{\mathbb{R}}$, $\mathscr{S}_{\mathbb{C}}$, respectively,
- $\mathscr{D}$ denotes the space of real compactly supported smooth functions on $\mathbb{R}^d$,

- $\mathscr{F}$, $\mathscr{F}_{-1}$ denote the Fourier and the inverse Fourier transformations

$$\mathscr{F}\varphi(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\langle x,y\rangle}\varphi(y)\,dy, \qquad \mathscr{F}_{-1}\varphi(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\langle x,y\rangle}\varphi(y)\,dy,$$

- $\mathscr{P}$ denotes the space of complex polynomials on $\mathbb{R}^d$,
- $\mathscr{L}(X,Y)$ denotes the space of linear continuous operators from X to Y,
- $\mathscr{L}_2(X,Y)$ denotes the space of γ-radonifying operators from X to Y (see Definition 5.1),
- $H^s = H^{s,2}(\mathbb{R}^d)$, $s \in \mathbb{R}$, is the scale of Sobolev spaces on $\mathbb{R}^d$,
- $\dot{B}^s_p$ denotes the homogeneous Besov space $\dot{B}^s_{p,2}(\mathbb{R}^d)$ (see Appendix A),
- if $1 \leqslant r \leqslant \infty$ then r' denotes the Hölder conjugate exponent $1 \leqslant r' \leqslant \infty$ such that $\frac{1}{r} + \frac{1}{r'} = 1$.

3. General assumptions

The following assumptions and hypotheses are imposed throughout this paper.

- We work on $\mathbb{R}^d$ for $d \geqslant 3$,
- μ is a finite symmetric measure on $\mathbb{R}^d$, i.e. $\mu(A) = \mu(-A)$ for every Borel $A \subseteq \mathbb{R}^d$,
- any filtration $(\mathscr{F}_t)$ of any probability space $(\Omega, \mathscr{F}, \mathbb{P})$ considered in this paper is assumed to be right-continuous and complete, i.e. $\mathscr{F}_t = \bigcap_{r>t} \mathscr{F}_r$, $t \geqslant 0$, and $F \in \mathscr{F}_0$ for every $\mathbb{P}$-negligible set $F \in \mathscr{F}$,
- $f, g : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz functions satisfying

$$f(0) = g(0) = 0, \qquad \left|f'(x)\right| \leqslant c\left(1 + |x|^{p-1}\right), \qquad \left|g'(x)\right| \leqslant c\left(1 + |x|^{\tilde{p}-1}\right) \tag{3.1}$$

for almost every $x \in \mathbb{R}$ where $p = \frac{d+2}{d-2}$, $\tilde{p} = \frac{d+1}{d-2}$ and $c \in (0, \infty)$.

4. Spatially homogeneous Wiener process

Following [25] (that we recommend as a good survey of properties and examples of spatially homogeneous Wiener processes), let μ be a finite symmetric measure on $\mathbb{R}^d$ (that we will call *a spectral measure*) and let $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ be a stochastic basis. A spatially homogeneous Wiener process with spectral measure μ may be introduced by two equivalent ways. The first one is to think of a centered Gaussian process $(\mathcal{W}(t,x)\colon\ t \geqslant 0,\ x \in \mathbb{R}^d)$ such that $(\mathcal{W}(t,x)\colon\ t \geqslant 0)$ is an $(\mathscr{F}_t)$-Wiener process for every $x \in \mathbb{R}^d$ and

$$\mathbb{E}\left\{\mathcal{W}(s,x)\mathcal{W}(t,y)\right\} = \min\{s,t\}\Gamma(x-y), \quad t, s \geqslant 0,\ x, y \in \mathbb{R}^d,$$

where $\Gamma : \mathbb{R}^d \to \mathbb{R}$ is the Fourier transform of $(2\pi)^{-\frac{d}{2}}\mu$. Another way is to consider an $\mathscr{S}'_{\mathbb{R}}$-valued $(\mathscr{F}_t)$-Wiener process satisfying

$$\mathbb{E}\left\{\left\langle W(s), \varphi_0\right\rangle\left\langle W(t), \varphi_1\right\rangle\right\} = \min\{s,t\}\langle\widehat{\varphi}_0, \widehat{\varphi}_1\rangle_{L^2(\mu)}, \quad t, s \geqslant 0,\ \varphi_0, \varphi_1 \in \mathscr{S}_{\mathbb{R}}.$$

The equivalent assignment between $\mathcal{W}$ and W is given by the formula (see e.g. page 190 in [25])

$$\left\langle W(t), \varphi\right\rangle = \int_{\mathbb{R}^d} \mathcal{W}(t,x)\varphi(x)\,dx, \quad t \geqslant 0,\ \varphi \in \mathscr{S}_{\mathbb{R}}.$$

The following proposition describes the reproducing kernel Hilbert space (RKHS) of a spatially homogeneous Wiener process and some of its properties. It is, in fact, an extension of Lemma 1 in [20].

Proposition 4.1. *Let W be a spatially homogeneous Wiener process with a finite spectral measure μ. Then the reproducing kernel Hilbert space of W (denoted by H_μ) is described as*

$$H_\mu = \big\{\widehat{\psi\mu}\colon\ \psi \in L^2_{\mathbb{C}}\big(\mathbb{R}^d, \mu\big),\ \overline{\psi(x)} = \psi(-x)\big\},$$
$$\langle\widehat{\psi_0\mu}, \widehat{\psi_1\mu}\rangle_{H_\mu} = \langle\psi_0, \psi_1\rangle_{L^2(\mu)},$$

H_μ is continuously embedded in the space of real continuous bounded functions on $\mathbb{R}^d$ and

$$\|\xi \mapsto h\xi\|_{\mathscr{L}_2(H_\mu, L^2(\mathbb{R}^d))} = \mathbf{c}\|h\|_{L^2(\mathbb{R}^d)}, \quad h \in L^2_{\mathbb{R}}\big(\mathbb{R}^d\big), \tag{4.1}$$

$$\|\xi \mapsto h\xi\|_{\mathscr{L}_2(H_\mu, H^{-1}(\mathbb{R}^d))} \leqslant c_\theta\|h\|_{L^\theta(\mathbb{R}^d)}, \quad h \in L^\theta_{\mathbb{R}}\big(\mathbb{R}^d\big), \tag{4.2}$$

hold for some constants $\mathbf{c} = c_d[\mu(\mathbb{R}^d)]^{\frac{1}{2}}$, $c_\theta \in \mathbb{R}_+$ and for any $\theta \in (\frac{2d}{d+2}, 2]$.

Proof. All claims were proved in Proposition 1.2 in [25] and in Lemma 1 in [20] except for (4.2). For let $\phi \in \mathscr{S}_{\mathbb{R}}$. Then

$$\sum_k \|\phi\xi_k\|^2_{H^{-1}} = \sum_k \int\limits_{\mathbb{R}^d} \frac{|\widehat{\phi\xi_k}(x)|^2}{1+|x|^2}\,dx = (2\pi)^{-d}\sum_k \int\limits_{\mathbb{R}^d} \frac{|\langle\widehat{\phi}(x-\cdot), \psi_k\rangle_{L^2(\mu)}|^2}{1+|x|^2}\,dx$$
$$\leqslant (2\pi)^{-d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \frac{|\widehat{\phi}(x-z)|^2}{1+|x|^2}\,dx\,\mu(dz) \leqslant c_1\|\widehat{\phi}\|^2_{L^{\theta'}} \leqslant c_2\|\phi\|^2_{L^\theta}$$

where $\xi_k = \widehat{\psi_k\mu}$, $k \geqslant 1$, is an ONB in H_μ. If $h \in L^\theta$, let $\phi_j \in \mathscr{S}_{\mathbb{R}}$ be such that $\lim_{j\to\infty}\|\theta_j - h\|_{L^\theta} = 0$. Then

$$\sum_k \|h\xi_k\|^2_{H^{-1}} \leqslant \liminf_{j\to\infty}\sum_k \|\phi_j\xi_k\|^2_{H^{-1}} \leqslant c_2\liminf_{j\to\infty}\|\phi_j\|^2_{L^\theta} = c_2\|h\|^2_{L^\theta}$$

by Fatou's lemma. □

5. Stochastic integration

Before we pass to stochastic integrals with respect to spatially homogeneous Wiener processes, we must recall the definition of γ-radonifying operators and 2-smooth Banach spaces.

Definition 5.1. If H is a real separable Hilbert space and X is a separable Banach space, a linear bounded operator $A : H \to X$ is γ-radonifying if there exists a centered Gaussian probability measure ν_A on X with the covariance AA^*. Such a measure is at most one, and in that case, we define

$$\|A\|^2_{\mathscr{L}_2(H,X)} = \int\limits_X \|x\|^2\,\nu_A(dx)$$

and denote by $\mathscr{L}_2(H, X)$ the set of γ-radonifying operators from H to X.

The vector space $(\mathscr{L}_2(H_\mu, X), \|\cdot\|_{\mathscr{L}_2(H_\mu,X)})$ is a separable Banach space (see [18] or [21]) and coincides with the space of Hilbert–Schmidt operators if X is a Hilbert space.

Definition 5.2. A Banach space X is 2-smooth provided that there exists a constant $c<\infty$ such that

$$\|x+y\|^2+\|x-y\|^2\leqslant 2\|x\|^2+2c\|y\|^2,\quad x,y\in X.$$

If W is a cylindrical $(\mathscr{F}_t)$-Wiener process on a real separable Hilbert space H then the stochastic integral $\int_0^T h\,dW$ can be constructed as a random variable in X provided that h is an $(\mathscr{F}_t)$-progressively measurable process with values in $\mathscr{L}_2(H,X)$ and

$$\int_0^T \big\|h(s)\big\|^2_{\mathscr{L}_2(H,X)}\,ds<\infty.$$

See [18,21] or [6] for details. We remark that $H=H_\mu$ if W is a spatially homogeneous Wiener process with spectral measure μ.

6. Solution

Let

- $(\Omega,\mathscr{F},(\mathscr{F}_t),\mathbb{P})$ be a stochastic basis,
- μ be a finite symmetric measure on $\mathbb{R}^d$,
- W be a spatially homogeneous $(\mathscr{F}_t)$-Wiener process with spectral measure μ,
- $0<\tau\leqslant\infty$ be an accessible $(\mathscr{F}_t)$-stopping time (i.e. there exists a sequence of $(\mathscr{F}_t)$-stopping times $\tau_n<\tau$ a.s. such that $\tau_n\uparrow\tau$ a.s.),
- $z=(u,v)$ be an $(\mathscr{F}_t)$-progressively measurable process with weakly continuous paths in $H^1_{\mathbb{R}}(\mathbb{R}^d)\oplus L^2_{\mathbb{R}}(\mathbb{R}^d)$ defined on the $(\mathscr{F}_t)$-progressively measurable set $\{(t,\omega):\ 0\leqslant t<\tau(\omega)\}$

and let

$$\big\langle u(\rho),\varphi\big\rangle=\big\langle u(0),\varphi\big\rangle+\int_0^\rho\big\langle v(s),\varphi\big\rangle ds,$$

$$\big\langle v(\rho),\varphi\big\rangle=\big\langle v(0),\varphi\big\rangle+\int_0^\rho\big\langle u(s),\Delta\varphi\big\rangle ds+\int_0^\rho\big\langle f\big(u(s)\big),\varphi\big\rangle ds+\int_0^\rho\big\langle g\big(u(s)\big)\,dW,\varphi\big\rangle \tag{6.1}$$

hold almost surely for every $\varphi\in\mathscr{D}$ and every $(\mathscr{F}_t)$-stopping time $\rho<\tau$ a.s. Here $\langle a,b\rangle=\int_{\mathbb{R}^d}a(x)b(x)\,dx$. The process z is then called *a solution* without further reference to (6.1) or (1.1).

Remark 6.1. Notice that $f(u)$ and $g(u)$ are $(\mathscr{F}_t)$-progressively measurable processes with locally bounded paths in $L^2(\mathbb{R}^d)+L^{\frac{2d}{d+2}}(\mathbb{R}^d)\subseteq H^{-1}(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)+L^{\frac{2d}{d+1}}(\mathbb{R}^d)\subseteq H^{-1}(\mathbb{R}^d)$, respectively. In fact, the estimations

$$\big\|f(u)\big\|_{H^{-1}}\leqslant c_1\big\|f(u)\big\|_{L^2+L^{\frac{2d}{d+2}}}\leqslant c_2\big(\|u\|_{L^2}+\|u\|^{\frac{d+2}{d-2}}_{L^{\frac{2d}{d-2}}}\big)\leqslant c_3\big(\|u\|_{H^1}+\|u\|^{\frac{d+2}{d-2}}_{H^1}\big),$$

$$\big\|\xi\mapsto g(u)\xi\big\|_{\mathscr{L}_2(H_\mu,H^{-1})}\leqslant c_4\big\|g(u)\big\|_{L^2+L^{\frac{2d}{d+1}}}\leqslant c_5\big(\|u\|_{L^2}+\|u\|^{\frac{d+1}{d-2}}_{L^{\frac{2d}{d-2}}}\big)\leqslant c_6\big(\|u\|_{H^1}+\|u\|^{\frac{d+1}{d-2}}_{H^1}\big)$$

obtained with the help of (4.2) yield not only that the integrals in (6.1) are convergent but also that (6.1) is equivalent with

$$u(\rho) = u(0) + \int_0^\rho v(s)\,ds, \tag{6.2}$$

$$v(\rho) = v(0) + \int_0^\rho \Delta u(s)\,ds + \int_0^\rho f\big(u(s)\big)\,ds + \int_0^\rho g\big(u(s)\big)\,dW \tag{6.3}$$

holding almost surely for every $(\mathscr{F}_t)$-stopping time $\rho < \tau$ a.s. The integral in (6.2) converges in L^2 and the integrals in (6.3) converge in H^{-1}.

As a consequence of norm continuity in $L^2 \oplus H^{-1}$ and weak continuity in $H^1 \oplus L^2$ of paths of a solution z, we get the following result by interpolation:

Theorem 6.2. *If $(z(t))_{t<\tau}$ is a solution and $\varepsilon > 0$ then $z \in C([0,\tau); H^{1-\varepsilon} \oplus H^{-\varepsilon})$ a.s.*

7. Main results

Let us start by a motivating example dealing with the subcritical case.

Example 7.1. It is known (see [22]) that if f and g have a subcritical growth, i.e. (3.1) holds for $p < \frac{d+2}{d-2}$, $\tilde{p} < \frac{d+1}{d-2}$, then there exists $\varepsilon \in [0, \frac{d}{d-1})$ such that

$$z \in L^q_{\mathrm{loc}}\big([0,\tau); \dot{B}_q^{\frac{1}{2}} \oplus \dot{B}_q^{-\frac{1}{2}}\big) \quad \text{a.s.} \tag{7.1}$$

where $q = 2\frac{d+1}{d-1}$, $z = (u, v)$ is a solution (understood in the sense of Section 6) and pathwise uniqueness holds for (6.1) among adapted processes with $H^1 \oplus L^2$-weakly continuous paths provided that

$$\int_{\mathbb{R}^d} \big(1 + |z|^\varepsilon\big)\,\mu(dz) < \infty. \tag{7.2}$$

We remark that the range $[0, \frac{d}{d-1})$ for ε is universal for all $p < \frac{d+2}{d-2}$ and $\tilde{p} < \frac{d+1}{d-2}$ but not optimal for particular p and $\tilde{p}$. For instance, existence of global solutions of (6.1) can be proved for many f, g satisfying (3.1) for $\tilde{p} \leqslant \frac{p+1}{2}$ (see e.g. [19]) and in this case pathwise uniqueness and (7.1) hold if (7.2) is satisfied for some $\varepsilon \in [0, 1)$ (see Remarks 9 and 36 in [22]).

Remark 7.2. Existence of solutions satisfying (7.1) for the critical nonlinearities, i.e. for f and g satisfying (3.1) for $p = \frac{d+2}{d-2}$, $\tilde{p} = \frac{d+1}{d-2}$, will be dealt with in a separate paper.

Motivated by Example 7.1, we return back to the general case of nonlinearities of the critical growth and finite spectral measure restricting however our attention to solutions in the class (7.1) where we will prove regularity (sharpening the result in Theorem 6.2) and an energy estimate.

Theorem 7.3 *(Regularity). Let the assumptions in Section* 3 *hold and let* (7.1) *be satisfied for a solution $(z(t))_{t<\tau}$. Then almost all paths of $z = (u, v)$ belong to $C([0,\tau); H^1 \oplus L^2)$.*

Theorem 7.4 *(Energy). Let the assumptions in Section* 3 *hold, let* (7.1) *be satisfied for a solution $(z(t))_{t<\tau}$, let $f_1, f_2, F_1, F_2 : \mathbb{R} \to \mathbb{R}$ satisfy*

- $f_1(0) = f_2(0) = F_1(0) = F_2(0) = F_1'(0) = F_2'(0) = 0$,
- $f = f_1 + f_2$,

- $f_1, f_2 \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$, $F_1, F_2 \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$, $f_1', F_1'' \in L^\infty(\mathbb{R})$,
- $|f_2'(r)| + |F_2''(r)| \leqslant c|r|^{\frac{4}{d-2}}$ *for a.e.* $r \in \mathbb{R}$ *and some* $c < \infty$.

Then almost all paths of $f_2(u)$ *and* $F_2'(u)$ *belong to* $L^{q'}_{\mathrm{loc}}([0,\tau); \dot{B}^{\frac{1}{2}}_{q'})$, *almost all paths of* $g(u)$ *belong to* $L^2_{\mathrm{loc}}([0,\tau); L^2)$ *and there exists* $M \in \mathscr{F}_0$, $\mathbb{P}(M) = 1$, *such that*

$$\mathbf{e}\big(z(t,\omega)\big) = \mathbf{e}\big(z(0,\omega)\big) + I(t,\omega) + \int_0^t \big\langle v(s,\omega), f_1\big(u(s,\omega)\big) + F_1'\big(u(s,\omega)\big)\big\rangle_{L^2}\, ds$$
$$+ \int_0^t \big\langle v(s,\omega), f_2\big(u(s,\omega)\big) + F_2'\big(u(s,\omega)\big)\big\rangle_{B_q^{-\frac{1}{2}} \times B_{q'}^{\frac{1}{2}}}\, ds + \frac{\mathbf{c}^2}{2} \int_0^t \big\|g\big(u(s,\omega)\big)\big\|^2_{L^2}\, ds \quad (7.3)$$

holds for every $\omega \in M$ *and every* $t \in [0, \tau(\omega))$ *where*

$$\mathbf{e}(z) = \frac{1}{2}\|\nabla u\|^2_{L^2} + \frac{1}{2}\|v\|^2_{L^2} + \int_{\mathbb{R}^d} F_1(u)\, dx + \int_{\mathbb{R}^d} F_2(u)\, dx, \quad z = (u,v) \in H^1 \oplus L^2, \quad (7.4)$$

and I *is a progressively measurable process with continuous paths defined on* $\{(t,\omega):\ t < \tau(\omega)\}$ *such that*

$$I(\rho) = \int_0^\rho \big\langle v(s), g\big(u(s)\big)\, dW\big\rangle_{L^2} \quad \text{a.s.}$$

holds for every $(\mathscr{F}_t)$*-stopping time* ρ *satisfying* $\rho < \tau$ *a.s.*

Concerning pathwise uniqueness of solutions of (6.1) in the critical case, we introduce a weaker hypothesis than (7.1), namely we say that (7.5) is satisfied for a solution $z = (u, v)$ defined on $[0, \tau)$ provided that

$$u \in L^s_{\mathrm{loc}}\big([0,\tau); L^s\big(\mathbb{R}^d\big)\big) \quad \text{a.s.} \quad (7.5)$$

where $s = 2\frac{d+1}{d-2}$. Indeed, (7.1) implies (7.5) by Lemma A.7.

Theorem 7.5 *(Pathwise uniqueness). Let the assumptions in Section* 3 *hold and let* $(z^i(t))_{t<\tau^i}$, $i = 1, 2$, *be two solutions with respect to a spatially homogeneous Wiener process* W *such that* $z^1(0) = z^2(0)$ *a.s. and* (7.5) *is satisfied for* z^1, z^2. *Then* $z^1(t) = z^2(t)$ *for every* $t < \tau^1 \wedge \tau^2$ *a.s.*

Theorem 7.6 *(Second component regularity). Let the assumptions in Section* 3 *hold and let* $(z(t))_{t<\tau}$ *be a solution with respect to a spatially homogeneous Wiener process* W *whose spectral measure* μ *satisfies*

$$\int_{\mathbb{R}^d} \big(1 + |y|\big)^{\frac{d}{d+1}+\varepsilon}\, \mu(dy) < \infty \quad (7.6)$$

for some $\varepsilon > 0$. *Then* z *satisfies* (7.1) *if and only if* $u \in L^q_{\mathrm{loc}}([0,\tau), \dot{B}^{\frac{1}{2}}_q)$ *a.s. where* $q = 2\frac{d+1}{d-1}$.

Remark 7.7. The condition (7.6) may be relaxed if g' has a subcritical growth. For instance, we can often prove existence of global solutions provided that $|g'(t)| \leqslant (1+|t|^{\frac{2}{d-2}})$ for a.e. $t \in \mathbb{R}$ (cf. [19]), and in this case (7.6) may be replaced by

$$\int_{\mathbb{R}^d} \big(1+|y|\big)^{\frac{d-1}{d+1}}\,\mu(dy) < \infty$$

in Theorem 7.6 (see Remark 9.2).

8. Proof of Theorems 7.3 and 7.4

The proof is divided into eleven steps.

Step 0. $H^1 \oplus L^2$-continuity of paths of z (i.e. Theorem 7.3) follows from (7.3), i.e. from Theorem 7.4. Indeed, choosing $F_1(x) = \frac{1}{2}|x|^2$ and $F_2(x) = 0$ in (7.3) we get that almost every path of $\|z\|_{H^1\oplus L^2}$ is continuous on $[0,\tau)$. Hence $z \in C([0,\tau); H^1 \oplus L^2)$ a.s. as weak $H^1 \oplus L^2$-continuity of paths of z was assumed.

Step 1. Let us prove Theorem 7.4. Paths of $f_2(u)$ and $F_2'(u)$ belong to $L^{q'}_{\mathrm{loc}}([0,\tau); \dot B^{\frac{1}{2}}_{q'})$ a.s. and paths of $g(u)$ belong to $L^2_{\mathrm{loc}}([0,\tau); L^2)$ a.s. by Lemma A.7. This is apparent for f_2 and F_2'. In case of g, find locally Lipschitz functions g_1, g_2 such that $g = g_1 + g_2$, $g_1(0) = g_2(0) = 0$, $\|g_1'\|_{L^\infty} < \infty$ and $|g_2'(x)| \leqslant c|x|^{\frac{3}{d-2}}$. Then $\|g_1(u)\|_{L^2} \leqslant c\|u\|_{L^2}$ and

$$\begin{aligned}
\big\|g_2(u)\big\|^2_{L^2((0,\tau_n);L^2)} &\leqslant c_1 \int_0^{\tau_n} \big\|u(t)\big\|^s_{L^s}\,dt \leqslant c_2 \int_0^{\tau_n} \big\|u(t)\big\|^q_{\dot B^{\frac{1}{2}}_q} \big\|u(t)\big\|^{\frac{s}{d-1}}_{H^1}\,dt \\
&\leqslant c_2 \|u\|^q_{L^q((0,\tau_n);\dot B^{\frac{1}{2}}_q)} \|u\|^{\frac{s}{d-1}}_{L^\infty((0,\tau_n);H^1)} < \infty \quad \text{a.s.}
\end{aligned}$$

for every $n \in \mathbb{N}$ where $s = 2\frac{d+1}{d-2}$ by Lemma A.7.

Step 2. Let $n \geqslant 1$ be fixed and let b_1 be a smooth compactly supported density on $\mathbb{R}^d$ and define $b_m(x) = m^d b_1(mx)$, $x \in \mathbb{R}^d$, and $z_m = (u_m, v_m)$ whose components are defined for $t \geqslant 0$ by the formulae

$$\begin{aligned}
u_m(t) &= b_m * u(0) + \int_0^t \mathbf{1}_{[s\leqslant\tau_n]} b_m * v(s)\,ds, \\
v_m(t) &= b_m * v(0) + \int_0^t \mathbf{1}_{[s\leqslant\tau_n]} \big\{\Delta b_m * u(s) + b_m * f\big(u(s)\big)\big\}\,ds \\
&\quad + \int_0^t \mathbf{1}_{[s\leqslant\tau_n]} b_m * \big[g\big(u(s)\big)\,dW\big].
\end{aligned} \tag{8.1}$$

Observing that $\xi \mapsto b_m * \xi : H^{-1} \to H^k$ is continuous for every $k \geqslant 0$ (by the Young inequality), the integrals in (8.1) converge in H^k for every $k \geqslant 0$ and z_m is an adapted process with continuous paths in $H^k \oplus H^k$, $k \geqslant 0$. Moreover, by (6.2), (6.3) and by continuity of paths,

$$z_m(t) = b_m * z(t), \quad t \in [0,\tau_n], \text{ a.s.} \tag{8.2}$$

Step 3. Assume that $h:\mathbb{R}\to\mathbb{R}$ is smooth, $h(0)=h'(0)=0$ and $k>\frac{d}{2}$, i.e. $H^k\subseteq C_b$ continuously where C_b is the space of bounded continuous functions on $\mathbb{R}^d$ equipped with the supremal norm. The function

$$\mathbf{e}_h(z)=\frac{1}{2}\|\nabla u\|_{L^2}^2+\frac{1}{2}\|v\|_{L^2}^2+\int\limits_{\mathbb{R}^d} h\big(u(x)\big)\,dx,\quad z=(u,v)\in H^k\oplus H^k,$$

is twice continuously differentiable on $H^k\oplus H^k$ and

$$\mathbf{e}_h'(z)a=\langle\nabla u,\nabla a_1\rangle_{L^2}+\langle v,a_2\rangle_{L^2}+\int\limits_{\mathbb{R}^d} h'\big(u(x)\big)a_1(x)\,dx,$$

$$\mathbf{e}_h''(z)(a,w)=\langle\nabla w_1,\nabla a_1\rangle_{L^2}+\langle w_2,a_2\rangle_{L^2}+\int\limits_{\mathbb{R}^d} h''\big(u(x)\big)a_1(x)w_1(x)\,dx$$

hold for $z=(u,v)$, $a=(a_1,a_2)$, $w=(w_1,w_2)\in H^k\oplus H^k$. The Itô formula (see e.g. Theorem 4.17 in [6]) now yields

$$\begin{aligned}\mathbf{e}_h\big(z_m(t)\big)=\mathbf{e}_h\big(z_m(0)\big)&+\int_0^{t\wedge\tau_n}\big\langle v_m(s),b_m*\big[g\big(u(s)\big)\,dW\big]\big\rangle_{L^2}\\&+\int_0^{t\wedge\tau_n}\big\langle\nabla u_m(s),\nabla b_m*v(s)\big\rangle_{L^2}\,ds+\int_0^{t\wedge\tau_n}\big\langle v_m(s),\Delta b_m*u(s)\big\rangle_{L^2}\,ds\\&+\int_0^{t\wedge\tau_n}\big\langle v_m(s),b_m*f\big(u(s)\big)\big\rangle_{L^2}\,ds+\int_0^{t\wedge\tau_n}\big\langle b_m*v(s),h'\big(u_m(s)\big)\big\rangle_{L^2}\,ds\\&+\frac{1}{2}\sum_k\int_0^{t\wedge\tau_n}\big\|b_m*\big[g\big(u(s)\big)e_k\big]\big\|_{L^2}^2\,ds,\quad t\geqslant 0,\end{aligned}$$

almost surely where (e_k) is any ONB in H_μ. Observe that, if $s\leqslant\tau_n$, then, almost surely,

$$\begin{aligned}\big\langle\nabla u_m(s),\nabla b_m*v(s)\big\rangle_{L^2}&=\big\langle\nabla b_m*u(s),\nabla b_m*v(s)\big\rangle_{L^2}\\&=-\big\langle\Delta b_m*u(s),b_m*v(s)\big\rangle_{L^2}\\&=-\big\langle\Delta b_m*u(s),v_m(s)\big\rangle_{L^2}\end{aligned}$$

by (8.2) and the integration by parts formula, so

$$\begin{aligned}\mathbf{e}_h\big(z_m(t)\big)=\mathbf{e}_h\big(z_m(0)\big)&+\int_0^{t\wedge\tau_n}\big\langle v_m(s),b_m*\big[g\big(u(s)\big)\,dW\big]\big\rangle_{L^2}\\&+\int_0^{t\wedge\tau_n}\big\langle v_m(s),b_m*f\big(u(s)\big)\big\rangle_{L^2}\,ds+\int_0^{t\wedge\tau_n}\big\langle b_m*v(s),h'\big(u_m(s)\big)\big\rangle_{L^2}\,ds\end{aligned}$$

$$+\frac{1}{2}\sum_k \int_0^{t\wedge\tau_n} \|b_m * [g(u(s))e_k]\|_{L^2}^2\,ds, \quad t\geqslant 0, \tag{8.3}$$

almost surely.

Step 4. Formula (8.3) holds also for $h\in W^{2,\infty}_{\rm loc}(\mathbb{R})$, $h(0)=h'(0)=0$ since, if δ is a smooth compactly supported density on $\mathbb{R}$ and $\delta_j(x)=j\delta(jx)$, the functions

$$h_j(x)=\delta_j * h(x) - x\delta_j * h'(0) - \delta_j * h(0), \quad x\in\mathbb{R},\ j\geqslant 1,$$

satisfy $h_j(0)=h_j'(0)=0$ and $h_j\to h$, $h_j'\to h'$ uniformly on bounded sets in $\mathbb{R}$ and $\sup_j |h_j''|$ is a locally bounded function on $\mathbb{R}$. In particular,

$$\left|h_j(x)-h(x)\right|\leqslant C_R|x|^2, \qquad \left|h_j'(x)-h'(x)\right|\leqslant C_R|x|, \quad |x|\leqslant R,\ j\geqslant 1.$$

So, if we realize that $\|u_m\|_{L^2(\mathbb{R}^d)\cap C_b(\mathbb{R}^d)}$ is uniformly bounded on every interval $[0,T]$, $h_j(u_m(t))\to h(u_m(t))$ in $L^1(\mathbb{R}^d)$ a.s. for every $t\geqslant 0$, $h_j'(u_m(t))\to h'(u_m(t))$ in $L^2(\mathbb{R}^d)$ a.s. for every $t\geqslant 0$ and

$$\sup\left\{\left\|h_j(u_m(t))\right\|_{L^1(\mathbb{R}^d)} + \left\|h_j'(u_m(t))\right\|_{L^2(\mathbb{R}^d)}\colon j\geqslant 1,\ t\leqslant T\right\}<\infty \quad \text{a.s.}$$

So we may pass to the limit in (8.3) for h_j arriving to (8.3) for h.

Step 5. Putting $h=F_1+F_2$ in (8.3) and using (8.2) and Remark A.4, we obtain

$$\begin{aligned}\mathbf{e}(b_m * z(t\wedge\tau_n)) &= \mathbf{e}(b_m * z(0)) + \int_0^{t\wedge\tau_n} \langle b_m * v(s), b_m * [g(u(s))\,dW]\rangle_{L^2} \\ &\quad + \int_0^{t\wedge\tau_n} \langle b_m * v(s), b_m * f_1(u(s)) + F_1'(b_m * u(s))\rangle_{L^2}\,ds \\ &\quad + \int_0^{t\wedge\tau_n} \langle b_m * v(s), b_m * f_2(u(s)) + F_2'(b_m * u(s))\rangle_{\dot B_q^{-\frac12}\times \dot B_{q'}^{\frac12}}\,ds \\ &\quad + \frac{1}{2}\sum_k \int_0^{t\wedge\tau_n} \|b_m * [g(u(s))e_k]\|_{L^2}^2\,ds, \quad t\geqslant 0,\end{aligned} \tag{8.4}$$

almost surely. This formula is correct since $(u,v)\in \dot B_q^{\frac12}\oplus \dot B_q^{-\frac12}$ for almost all (t,ω) by (7.1) and $b_m *\colon \dot B_q^{\pm\frac12}\to \dot B_q^{\pm\frac12}$ by Lemma A.5. Also $b_m * f_2(u)+F_2'(b_m * u)\in \dot B_{q'}^{\frac12}$ for almost all (t,ω) by Lemma A.7. Let us deal with all terms in (8.4).

Step 6. The term $b_m * z(t\wedge\tau_n)\to z(t\wedge\tau_n)$ in $H^1\oplus L^2$ for every (t,ω) so $\mathbf{e}(b_m * z(t\wedge\tau_n))\to \mathbf{e}(z(t\wedge\tau_n))$ and $\mathbf{e}(b_m * z(0))\to \mathbf{e}(z(0))$ for every (t,ω).

Step 7. It holds that

$$\left|\langle b_m * v, b_m * [g(u)e_k]\rangle_{L^2}\right|^2 \leqslant \|v\|_{L^2}^2\left\|g(u)e_k\right\|_{L^2}^2$$

and

$$\sum_k \int_0^{\tau_n} \|v(s)\|_{L^2}^2 \|g(u(s))e_k\|_{L^2}^2\, ds = \mathbf{c}^2 \int_0^{\tau_n} \|v(s)\|_{L^2}^2 \|g(u(s))\|_{L^2}^2\, ds < \infty \quad \text{a.s.}$$

by (4.1) since paths of v are locally bounded in L^2 and paths of $g(u)$ are locally L^2-integrable in L^2 almost surely as shown in Step 1. Thus, by the Lebesgue dominated convergence theorem,

$$\lim_{m\to\infty} \sum_k \int_0^{\tau_n} \left| \langle b_m * v(s), b_m * [g(u(s))e_k] \rangle_{L^2} - \langle v(s), g(u(s))e_k \rangle_{L^2} \right|^2 ds = 0 \quad \text{a.s.},$$

so

$$\lim_{m\to\infty} \left\{ \sup_{0\leqslant t\leqslant T} \left| \int_0^{t\wedge\tau_n} \langle b_m * v(s), b_m * [g(u(s))\, dW] \rangle_{L^2} - \int_0^{t\wedge\tau_n} \langle v(s), g(u(s))\, dW \rangle_{L^2} \right| \right\} = 0$$

in probability for every $T > 0$ by Proposition 4.1 in [21].

Step 8. It holds that

$$\left| \langle b_m * v, b_m * f_1(u) + F_1'(b_m * u) \rangle_{L^2} \right| \leqslant c \|v\|_{L^2} \|u\|_{L^2}$$

as f_1, F_1' are Lipschitz functions, hence, by the Lebesgue dominated convergence theorem,

$$\lim_{m\to\infty} \int_0^{\tau_n} \left| \langle b_m * v(s), b_m * f_1(u(s)) + F_1'(b_m * u(s)) \rangle_{L^2} - \langle v(s), f_1(u(s)) + F_1'(u(s)) \rangle_{L^2} \right| ds = 0$$

on Ω since paths of v and u are locally bounded in L^2.

Step 9. It holds that

$$\begin{aligned} \left| \langle b_m * v, b_m * f_2(u) + F_2'(b_m * u) \rangle_{\dot{B}_q^{-\frac{1}{2}} \times \dot{B}_{q'}^{\frac{1}{2}}} \right| &\leqslant \tilde{c} \|v\|_{\dot{B}_q^{-\frac{1}{2}}} \left(\|f_2(u)\|_{\dot{B}_{q'}^{\frac{1}{2}}} + \|F_2'(b_m * u)\|_{\dot{B}_{q'}^{\frac{1}{2}}} \right) \\ &\leqslant c \|v\|_{\dot{B}_q^{-\frac{1}{2}}} \|u\|_{\dot{B}_q^{\frac{1}{2}}}^{q-1} \|u\|_{H^1}^{\frac{4}{(d-1)(d-2)}} \end{aligned}$$

by Corollary A.3, Lemmas A.5 and A.7, and

$$\int_0^{\tau_n} \|v(s)\|_{\dot{B}_q^{-\frac{1}{2}}} \|u(s)\|_{\dot{B}_q^{\frac{1}{2}}}^{q-1} \|u(s)\|_{H^1}^{\frac{4}{(d-1)(d-2)}} ds \leqslant \|v\|_{L^q((0,\tau_n);\dot{B}_q^{-\frac{1}{2}})} \|u\|_{L^q((0,\tau_n);\dot{B}_q^{\frac{1}{2}})}^{q-1} \|u\|_{C([0,\tau_n];H^1)}$$

which is finite almost surely by (7.1). Next

$$\lim_{m\to\infty} \left[\|b_m * v - v\|_{\dot{B}_q^{-\frac{1}{2}}} + \|b_m * f_2(u) - f_2(u)\|_{\dot{B}_q^{-\frac{1}{2}}} \right] = 0$$

for almost all (t,ω) by Lemmas A.5 and A.7. It also holds that

$$\lim_{m\to\infty}\left\|F_2'(b_m*u)-F_2'(u)\right\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)}=0,$$

so

$$\lim_{m\to\infty}\left\langle\theta, F_2'(b_m*u)\right\rangle_{\dot{B}_q^{-\frac12}\times\dot{B}_{q'}^{\frac12}}=\left\langle\theta, F_2'(u)\right\rangle_{\dot{B}_q^{-\frac12}\times\dot{B}_{q'}^{\frac12}}$$

holds for every $\theta\in\mathscr{Z}_{\mathbb{C}}$ and since $F_2'(b_m*u)$ is bounded in $\dot{B}_{q'}^{\frac12}$ by Lemma A.7 for almost all (t,ω) and $\mathscr{Z}_{\mathbb{C}}$ is dense in $\dot{B}_q^{-\frac12}$ by Proposition A.1,

$$\lim_{m\to\infty}\left\langle v, F_2'(b_m*u)\right\rangle_{\dot{B}_q^{-\frac12}\times\dot{B}_{q'}^{\frac12}}=\left\langle v, F_2'(u)\right\rangle_{\dot{B}_q^{-\frac12}\times\dot{B}_{q'}^{\frac12}}$$

holds for almost all (t,ω). We have thus proved that

$$\lim_{m\to\infty}\int_0^{\tau_n}\left|\left\langle b_m*v, b_m*f_2(u)+F_2'(b_m*u)\right\rangle_{\dot{B}_q^{-\frac12}\times\dot{B}_{q'}^{\frac12}}-\left\langle v, f_2(u)+F_2'(u)\right\rangle_{\dot{B}_q^{-\frac12}\times\dot{B}_{q'}^{\frac12}}\right|ds=0$$

almost surely by the Lebesgue dominated convergence theorem.

Step 10. The inequality $\|b_m*[g(u)e_k]\|_{L^2}^2\leqslant\|g(u)e_k\|_{L^2}^2$ holds and

$$\sum_k\int_0^{\tau_n}\left\|g\big(u(s)\big)e_k\right\|_{L^2}^2\,ds=\mathbf{c}^2\int_0^{\tau_n}\left\|g(u)\right\|_{L^2}^2\,ds<\infty\quad\text{a.s.}$$

by (4.1) and Step 1, so

$$\lim_{m\to\infty}\sum_k\int_0^{\tau_n}\left|\left\|b_m*\left[g\big(u(s)\big)e_k\right]\right\|_{L^2}^2-\left\|g\big(u(s)\big)e_k\right\|_{L^2}^2\right|ds=0$$

almost surely by the Lebesgue dominated convergence theorem. The equality (7.3) is finally proved.

9. Proofs of Theorems 7.5 and 7.6

Let us introduce operators

$$T_t=\begin{pmatrix}\dot{K}_t & K_t\\ \Delta K_t & \dot{K}_t\end{pmatrix},\quad t\in\mathbb{R},\tag{9.1}$$

where $\dot{K}_t\xi=\mathscr{F}_{-1}(\frac{\sin(t|x|)}{|x|}\cdot\mathscr{F}\xi)$ and $K_t\xi=\mathscr{F}_{-1}(\cos(t|x|)\cdot\mathscr{F}\xi)$ for $\xi\in\mathscr{Z}'$. By e.g. Proposition 20 in [22], the linear operators (T_t) form a C_0-group on $\mathscr{Z}'\oplus\mathscr{Z}'$ as well as on $L^2\oplus H^{-1}$ and the infinitesimal generator is

$$\begin{pmatrix}0 & I\\ \Delta & 0\end{pmatrix}$$

whose domain is, in the latter case, $H^1\oplus L^2$.

Lemma 9.1. *Let $z = (u, v)$ be a solution defined on $[0, \tau)$ and let ρ be an $(\mathscr{F}_t)$-stopping time such that $\rho < \tau$ a.s. Define*

$$z_\rho(t) = T_t z(0) + \int_0^t \mathbf{1}_{[0,\rho)}(s) T_{t-s} \begin{pmatrix} 0 \\ f(u(s)) \end{pmatrix} ds + \int_0^t \mathbf{1}_{[0,\rho)}(s) T_{t-s} \begin{pmatrix} 0 \\ g(u(s)) \end{pmatrix} dW$$

for $t \geqslant 0$. Then $z(t) = z_\rho(t)$ for $t \in [0, \rho]$ a.s.

Proof. The integrals converge in $L^2 \oplus H^{-1}$ by Remark 6.1 and

$$u_\rho(t) = u(0) + \int_0^t v_\rho(s)\, ds, \tag{9.2}$$

$$v_\rho(t) = v(0) + \Delta \int_0^t u_\rho(s)\, ds + \int_0^{t\wedge\rho} f\big(u(s)\big)\, ds + \int_0^{t\wedge\rho} g\big(u(s)\big)\, dW \tag{9.3}$$

holds a.s. for every $t \geqslant 0$ where the integral in (9.2) converges in L^2 and the integrals in (9.3) converge in H^{-1}, by the Chojnowska-Michalik theorem (see [4] or Theorem 12 in [21]). Taking the difference in (6.2), (6.3) and (9.2), (9.3), we get

$$u_\rho(t) - u(t) = \int_0^t v_\rho(s) - v(s)\, ds, \qquad v_\rho(t) - v(t) = \Delta \int_0^t u_\rho(s) - u(s)\, ds$$

for every $t \leqslant \rho$ almost surely, hence $z(t) = z_\rho(t)$ for $t \in [0, \rho]$ a.s. by uniqueness of the deterministic wave equation $\psi_{tt} = \Delta\psi$ in the Hilbert space $L^2 \oplus H^{-1}$. □

9.1. Proof of Theorem 7.5

Introduce $(\mathscr{F}_t)$-stopping times

$$\sigma_n^0 = n \wedge \tau_n^1 \wedge \tau_n^2,$$
$$\sigma_n^1 = \sigma_n^0 \wedge \inf\big\{t < \tau^1\colon \big\|z^1(t)\big\|_{H^1\oplus L^2} > n\big\} \wedge \inf\big\{t < \tau^2\colon \big\|z^2(t)\big\|_{H^1\oplus L^2} > n\big\},$$
$$\sigma_n^2 = \sigma_n^1 \wedge \inf\big\{t < \tau^1 \wedge \tau^2\colon \big\|u^1(t) - u^2(t)\big\|_{H^1} > 0\big\},$$
$$t^* = \inf\big\{t < \tau^1 \wedge \tau^2\colon \big\|u^1(t) - u^2(t)\big\|_{H^1} > 0\big\}$$

with the convention $\inf\emptyset = \infty$, and the process

$$V(t) = \int_0^t \mathbf{1}_{[\sigma_n^2,\sigma_n^1]}(r)\big(1 + \big\|u^1(r)\big\|_{L^s}^s + \big\|u^2(r)\big\|_{L^s}^s\big)\, dr, \quad t \geqslant 0,$$

where $s = 2\frac{d+1}{d-2}$. The adapted process V has continuous paths almost surely by (7.5), for we may define an $(\mathscr{F}_t)$-stopping time

$$\rho_\varepsilon = \sigma_n^1 \wedge \inf\big\{t \geqslant 0\colon V(t) \geqslant \varepsilon\big\}, \quad \varepsilon \in (0, 1).$$

We have, for $t \in [0, n]$,

$$\begin{aligned}
&\mathbb{E}\left\|u^1_{\rho_\varepsilon}(t) - u^2_{\rho_\varepsilon}(t)\right\|^\theta_{L^q} \\
&\quad\leqslant c\mathbb{E}\left[\int_0^t \mathbf{1}_{[0,\rho_\varepsilon]}(r)\left\|K_{t-r}\left(f\left(u^1(r)\right) - f\left(u^2(r)\right)\right)\right\|_{L^q}\, dr\right]^\theta \\
&\qquad + c\mathbb{E}\left[\int_0^t \mathbf{1}_{[0,\rho_\varepsilon]}(r)\left\|K_{t-r}\circ\left(g\left(u^1(r)\right) - g\left(u^2(r)\right)\right)\right\|^2_{\mathscr{L}_2(H_\mu, L^q)}\, dr\right]^{\frac{\theta}{2}} \\
&\quad\leqslant c\mathbb{E}\left[\int_0^t \mathbf{1}_{[0,\rho_\varepsilon]}(r)(t-r)^{-\frac{d-1}{d+1}}\left\|u^1(r) - u^2(r)\right\|_{L^q}\left(1 + \left\|u^1(r)\right\|^{\frac{4}{d-2}}_{L^s} + \left\|u^2(r)\right\|^{\frac{4}{d-2}}_{L^s}\right) dr\right]^\theta \\
&\qquad + c\mathbb{E}\left[\int_0^t \mathbf{1}_{[0,\rho_\varepsilon]}(r)(t-r)^{-\frac{2}{s}}\left\|u^1(r) - u^2(r)\right\|^2_{L^q}\left(1 + \left\|u^1(r)\right\|^{\frac{6}{d-2}}_{L^s} + \left\|u^2(r)\right\|^{\frac{6}{d-2}}_{L^s}\right) dr\right]^{\frac{\theta}{2}} \\
&\quad= c\mathbb{E}\left[\int_0^t \mathbf{1}_{[\sigma^2_n,\rho_\varepsilon]}(r)(t-r)^{-\frac{d-1}{d+1}}\left\|u^1(r) - u^2(r)\right\|_{L^q}\left(1 + \left\|u^1(r)\right\|^{\frac{4}{d-2}}_{L^s} + \left\|u^2(r)\right\|^{\frac{4}{d-2}}_{L^s}\right) dr\right]^\theta \\
&\qquad + c\mathbb{E}\left[\int_0^t \mathbf{1}_{[\sigma^2_n,\rho_\varepsilon]}(r)(t-r)^{-\frac{2}{s}}\left\|u^1(r) - u^2(r)\right\|^2_{L^q}\left(1 + \left\|u^1(r)\right\|^{\frac{6}{d-2}}_{L^s} + \left\|u^2(r)\right\|^{\frac{6}{d-2}}_{L^s}\right) dr\right]^{\frac{\theta}{2}}
\end{aligned}$$

by the Burkholder inequality (see e.g. [21]), Lemma A.9 and Proposition A.11. Here $c = c_{d,n,f,g,\mu,\theta}$ may differ from step to another. Choosing $\theta \in (s, \infty)$ we can use Lemma A.12 to obtain

$$\begin{aligned}
\mathbb{E}\int_0^n \left\|u^1_{\rho_\varepsilon}(t) - u^2_{\rho_\varepsilon}(t)\right\|^\theta_{L^q} dt &\leqslant c\mathbb{E}\left[\int_0^\infty \mathbf{1}_{[\sigma^2_n,\rho_\varepsilon]}(r)\left\|u^1(r) - u^2(r)\right\|^\theta_{L^q} dr\right]\left[V(\rho_\varepsilon)^{\frac{2\theta}{d+1}} + V(\rho_\varepsilon)^{\frac{3\theta}{2(d+1)}}\right] \\
&\leqslant c\varepsilon^{\frac{3\theta}{2(d+1)}}\mathbb{E}\left[\int_0^\infty \mathbf{1}_{[\sigma^2_n,\rho_\varepsilon]}(r)\left\|u^1(r) - u^2(r)\right\|^\theta_{L^q} dr\right],
\end{aligned}$$

so

$$\mathbb{E}\int_{\sigma^2_n}^{\rho_\varepsilon} \left\|u^1(r) - u^2(r)\right\|^\theta_{L^q} dr = 0 \tag{9.4}$$

for small ε as $u^i = u^i_{\rho_\varepsilon}$ on $[0, \rho_\varepsilon]$ by Lemma 9.1. Hence $u^1 = u^2$ on $[0, \rho_\varepsilon]$ by (9.4) which yields $t^* \geqslant \rho_\varepsilon$, consequently $\sigma^1_n \leqslant t^*$ (otherwise, by definition, $\sigma^2_n = t^*$ and $V(\rho_\varepsilon) = \varepsilon$ so it would hold that $V \equiv 0$ on $[0, \sigma^2_n] = [0, t^*]$ hence $\rho_\varepsilon > t^*$). But $\sigma^1_n \uparrow \tau^1 \wedge \tau^2$.

9.2. Proof of Theorem 7.6

The process $z^n=(u^n,v^n)=\mathbf{1}_{[\|z(0)\|_{H^1\oplus L^2}\leqslant n]}z$ defined on $[0,\tau)$ is a solution of (6.1) since $f(0)=g(0)=0$. Introduce the $(\mathscr{F}_t)$-stopping times

$$\sigma_n^0=\inf\{t<\tau\colon\ \|z^n(t)\|_{H^1\oplus L^2}>n\},$$

$$\sigma_n^1=\inf\left\{t<\tau\colon\ \int_0^t\|u^n(r)\|^q_{\dot B_q^{\frac12}}\,dr>n\right\},$$

$$\rho_n=n\wedge\tau_n\wedge\sigma_n^0\wedge\sigma_n^1$$

with the convention $\inf\emptyset=\infty$, and a process $Z^n=(U^n,V^n)$

$$Z^n(t)=T_tz^n(0)+\int_0^t\mathbf{1}_{[0,\rho_n)}(s)T_{t-s}\begin{pmatrix}0\\ f(u^n(s))\end{pmatrix}ds+\int_0^t\mathbf{1}_{[0,\rho_n)}(s)T_{t-s}\begin{pmatrix}0\\ g(u^n(s))\end{pmatrix}dW$$

for $t\geqslant 0$. We obtain, with the constant c changing from line to line and depending on d, n, q, μ, f and g, the inequality

$$\begin{aligned}\mathbb{E}\int_0^n\|V^n(t)\|^q_{\dot B_q^{-\frac12}}\,dt&\leqslant c\mathbb{E}\int_0^n\|T_tz^n(0)\|^q_{\dot B_q^{\frac12}\oplus\dot B_q^{-\frac12}}\,dt\\&\quad+c\mathbb{E}\int_0^n\left[\int_0^t\mathbf{1}_{[0,\rho_n)}(r)\big\|\dot K_{t-r}\big(f\big(u^n(r)\big)\big)\big\|_{\dot B_q^{-\frac12}}\,dr\right]^q dt\\&\quad+c\mathbb{E}\int_0^n\left[\int_0^t\mathbf{1}_{[0,\rho_n)}(r)\big\|\dot K_{t-r}\circ g\big(u^n(r)\big)\big\|^2_{\mathscr{L}_2(H_\mu,\dot B_q^{-\frac12})}\,dr\right]^{\frac q2}dt\\&=I_1+I_2+I_3\end{aligned}$$

by the Burkholder inequality (see e.g. (5.1) in [21]) as $\dot B_q^{-\frac12}$ is a 2-smooth Banach space by Proposition A.1. Decomposing $f=f_0+f_1$ and $g=g_0+g_1$ so that f_i, g_i are locally Lipschitz functions for $i=1,2$ such that f_0, g_0 are compactly supported and $f=f_0$, $g=g_0$ on a neighborhood around $0\in\mathbb{R}$. Then

$$I_1\leqslant c\mathbb{E}\|z^n(0)\|^q_{H^1\oplus L^2}\leqslant cn^q$$

by Proposition A.10. If $r<\rho_n$ then

$$\big\|\dot K_{t-r}\big(f_0\big(u^n(r)\big)\big)\big\|_{\dot B_q^{-\frac12}}\leqslant c\|u^n(r)\|_{H^1}\leqslant cn,$$

$$\begin{aligned}\big\|\dot K_{t-r}\big(f_1\big(u^n(r)\big)\big)\big\|_{\dot B_q^{-\frac12}}&\leqslant c|t-r|^{-\frac{d-1}{d+1}}\|u^n(r)\|^{q-1}_{\dot B_q^{\frac12}}\|u^n(r)\|^{\frac{4}{(d-2)(d-1)}}_{L^{\frac{2d}{d-2}}}\\&\leqslant c|t-r|^{-\frac{d-1}{d+1}}\|u^n(r)\|^{q-1}_{\dot B_q^{\frac12}}\end{aligned}$$

by Lemma A.8. Hence

$$I_2 \leqslant c + c\mathbb{E}\left(\int_0^{\rho_n} \|u^n(r)\|^q_{\dot{B}_q^{\frac{1}{2}}}\,dr\right)^{q-1} \leqslant c$$

by the Hardy–Littlewood–Sobolev inequality. If $r < \rho_n$ then

$$\begin{aligned}
&\|\dot{K}_{t-r} \circ g_0(u^n(r))\|^2_{\mathscr{L}_2(H_\mu, \dot{B}_q^{-\frac{1}{2}})} \\
&\quad \leqslant c \int_{\mathbb{R}^d} \|\dot{K}_{t-r}(e^{i\langle y,\cdot\rangle} g_0(u^n(r)))\|^2_{\dot{B}_q^{-\frac{1}{2}}}\,\mu(dy) \\
&\quad \leqslant c \int_{\mathbb{R}^d} (1 + |y|^{\frac{2}{q}}) \|u^n(r)\|^2_{H^1}\,\mu(dy) \leqslant cn^2, \\
&\|\dot{K}_{t-r} \circ g_1(u^n(r))\|^2_{\mathscr{L}_2(H_\mu, \dot{B}_q^{-\frac{1}{2}})} \\
&\quad \leqslant c \int_{\mathbb{R}^d} \|\dot{K}_{t-r}(e^{i\langle y,\cdot\rangle} g_1(u^n(r)))\|^2_{\dot{B}_q^{-\frac{1}{2}}}\,\mu(dy) \\
&\quad \leqslant c \int_{\mathbb{R}^d} |t-r|^{-2w} \|u^n(r)\|^{2+q(1-2w)}_{\dot{B}_q^{\frac{1}{2}}} \|u^n(r)\|^{\frac{6}{d-2}-q(1-2w)}_{\frac{2d}{d-2}}\,\mu(dy) \\
&\qquad + c \int_{\mathbb{R}^d} |t-r|^{-2w} |y|^{2(\frac{1}{q}+\frac{w}{d-1})} \|u^n(r)\|^{q(1-\frac{2dw}{d-1})}_{\dot{B}_q^{\frac{1}{2}}} \|u^n(r)\|^{2\frac{d-1}{d-2}-q(1-\frac{2dw}{d-1})}_{\frac{2d}{d-2}}\,\mu(dy) \\
&\quad \leqslant c|t-r|^{-2w}\left[\|u^n(r)\|^{2+q(1-2w)}_{\dot{B}_q^{\frac{1}{2}}} + \|u^n(r)\|^{q(1-\frac{2dw}{d-1})}_{\dot{B}_q^{\frac{1}{2}}}\right]
\end{aligned} \tag{9.5}$$

by Proposition A.11 and Lemma A.8 where $\frac{1}{q} < w < \frac{d-1}{2d}$ is chosen so that $2(\frac{1}{q} + \frac{w}{d-1}) < \frac{d}{d+1} + \varepsilon$. Hence

$$\begin{aligned}
I_3 &\leqslant c + c\mathbb{E}\left[\int_0^{\rho_n} \|u^n(r)\|^q_{\dot{B}_q^{\frac{1}{2}}}\,dr\right]^{\frac{q}{2a}} + c\mathbb{E}\left[\int_0^{\rho_n} \|u^n(r)\|^{q\theta}_{\dot{B}_q^{\frac{1}{2}}}\,dr\right]^{\frac{q}{2a}} \\
&\leqslant c + c\mathbb{E}\left[\int_0^{\rho_n} \|u^n(r)\|^q_{\dot{B}_q^{\frac{1}{2}}}\,dr\right]^{\frac{q}{2a}} + c\mathbb{E}\left[\int_0^{\rho_n} \|u^n(r)\|^{q}_{\dot{B}_q^{\frac{1}{2}}}\,dr\right]^{\frac{q\theta}{2a}} \leqslant c
\end{aligned}$$

by the Hardy–Littlewood–Sobolev inequality where $a \in \mathbb{R}$ satisfies $2w + \frac{1}{a} = 1 + \frac{2}{q}$ and $\theta = \frac{q(1-\frac{2dw}{d-1})}{2+q(1-2w)} \in (0,1)$. This result with Lemma 9.1 imply that

$$\mathbb{E}\int_0^{\rho_n} \|v^n(r)\|^q_{\dot{B}_q^{-\frac{1}{2}}}\,dr < \infty$$

as $\rho_n \leqslant n$ and $v^n = V^n$ on $[0, \rho_n]$ a.s., hence

$$\mathbf{1}_{[\|z(0)\|_{H^1\oplus L^2}\leqslant n]}\int_0^{\rho_n} \|v(r)\|^q_{\dot{B}_q^{-\frac{1}{2}}}\,dr = \int_0^{\rho_n} \|v^n(r)\|^q_{\dot{B}_q^{-\frac{1}{2}}}\,dr < \infty \quad \text{a.s.}$$

and we get the result from the fact that $\lim \rho_n = \tau$ a.s.

Remark 9.2. As far as Remark 7.7 is concerned, the proof of Theorem 7.6 must be modified in the inequality (9.5) which goes, for $r < \rho_n$, as

$$\begin{aligned}\big\|\dot{K}_{t-r}\circ g_1\big(u^n(r)\big)\big\|^2_{\mathscr{L}_2(H_\mu,\dot{B}_q^{-\frac{1}{2}})} &\leqslant c\int_{\mathbb{R}^d} \big\|\dot{K}_{t-r}\big(e^{i\langle y,\cdot\rangle}g_1\big(u^n(r)\big)\big)\big\|^2_{\dot{B}_q^{-\frac{1}{2}}}\,\mu(dy)\\ &\leqslant c\int_{\mathbb{R}^d}\Big[\big\|u^n(r)\big\|^2_{\dot{B}_q^{\frac{1}{2}}}\big\|u^n(r)\big\|^{\frac{4}{d-2}}_{\frac{2d}{d-2}} + |y|^{\frac{2}{q}}\big\|u^n(r)\big\|^{\frac{2d}{d-2}}_{\frac{2d}{d-2}}\Big]\,\mu(dy)\\ &\leqslant c\big(1+\big\|u^n(r)\big\|^2_{\dot{B}_q^{\frac{1}{2}}}\big)\end{aligned}$$

by Proposition A.11 and Lemma A.8. Hence

$$I_3 \leqslant c + c\mathbb{E}\int_0^{\rho_n}\big\|u^n(r)\big\|^q_{\dot{B}_q^{\frac{1}{2}}}\,dr \leqslant c.$$

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Appendix A. Homogeneous Besov spaces

A.1. Basic properties

Let us introduce the vector space

$$\begin{aligned}\mathscr{Z} &= \big\{\varphi\in\mathscr{S}_{\mathbb{C}}\colon\ D^\alpha\widehat{\varphi}(0)=0,\ \alpha\in\mathbb{N}_0^d\big\}\\ &= \Big\{\varphi\in\mathscr{S}_{\mathbb{C}}\colon \int_{\mathbb{R}^d} x^\alpha\varphi(x)\,dx = 0,\ \alpha\in\mathbb{N}_0^d\Big\}\end{aligned}$$

that we equip with the topology of $\mathscr{S}_{\mathbb{C}}$, and let $\mathscr{Z}'$ be its dual space. It can be shown (see 5.1.2 in [30]) that every $T\in\mathscr{Z}'$ can be extended to a distribution $S\in\mathscr{S}'_{\mathbb{C}}$, i.e. $T\subseteq S$, and any other distribution $\tilde{S}\in\mathscr{S}_{\mathbb{C}}$ extends T if and only if $\tilde{S}-S$ is a polynomial. Hence $\mathscr{Z}'$ is isomorphic to $\mathscr{S}_{\mathbb{C}}/\mathscr{P}$ where $\mathscr{P}$ denotes the space of polynomials. Moreover, if $\phi_j\in C^\infty(\mathbb{R}^d)$ are real functions with supports in $\{2^{j-1}<|x|<2^{j+1}\}$ and $\sum_{j\in\mathbb{Z}}\phi_j = 1$ on $\mathbb{R}^d\setminus\{0\}$ then we define

$$\|T\|_{\dot{B}^s_r} = \left\{ \sum_{j\in\mathbb{Z}} 2^{2js} \|T * \mathscr{F}^{-1}\phi_j\|^2_{L^r(\mathbb{R}^d)} \right\}^{\frac{1}{2}}, \quad T \in \mathscr{Z}',$$

$$\dot{B}^s_r = \{T \in \mathscr{Z}' \colon \|T\|_{\dot{B}^s_r} < \infty\} \tag{A.1}$$

for $s \in \mathbb{R}$, $1 \leqslant r \leqslant \infty$ where $\mathscr{F}$ and $\mathscr{F}^{-1}$ are the Fourier and the inverse Fourier transformations, respectively.

Proposition A.1. *Let $s \in \mathbb{R}$. The space $(\dot{B}^s_r, \|\cdot\|_{\dot{B}^s_r})$ is a Banach space for $1 \leqslant r \leqslant \infty$ which is separable if $1 \leqslant r < \infty$, reflexive if $1 < r < \infty$ and 2-smooth if $2 \leqslant r < \infty$. Moreover, $\mathscr{Z}$ is dense in $\dot{B}^s_r$ and $\dot{B}^{-s}_{r'} = (\mathscr{Z}, \|\cdot\|_{\dot{B}^s_r})^*$ if $1 \leqslant r < \infty$.*

Proof. See Theorem 5.1.5 in [30] for completeness, density of $\mathscr{Z}$ and separability. The space $\dot{B}^s_r$ is reflexive, resp. 2-smooth as it is isomorphic with a closed subspace of $l_2(\mathbb{Z}; L^r)$ and the duality between $\dot{B}^{-s}_{r'} = (\dot{B}^s_r)^*$ can be proved analogously to the proof of Theorem 2.11.2 in [30]. □

Remark A.2. The space $\mathscr{Z}$ is dense in L^r for $1 < r < \infty$. See Theorems 5.2.3/1 and 5.1.5 in [30].

Corollary A.3. *Let $1 \leqslant r < \infty$. The bilinear form defined by the dual pairing*

$$\dot{B}^{-s}_{r'} \times \mathscr{Z} \to \mathbb{C} : (S, \varphi) \mapsto \langle S, \varphi \rangle$$

extends in a unique way to a continuous bilinear form

$$\dot{B}^{-s}_{r'} \times \dot{B}^s_r \to \mathbb{C} : (S, T) \mapsto \langle S, T \rangle_{\dot{B}^{-s}_{r'} \times \dot{B}^s_r}.$$

Proof. This is a consequence of the duality $\dot{B}^{-s}_{r'} = (\mathscr{Z}, \|\cdot\|_{\dot{B}^s_r})^*$ (see Proposition A.1). □

Remark A.4. If $1 \leqslant r < \infty$, $h_1 \in L^2 \cap \dot{B}^{-s}_{r'}$, $h_2 \in L^2 \cap \dot{B}^s_r$ then

$$\langle h_1, h_2 \rangle_{\dot{B}^{-s}_{r'} \times \dot{B}^s_r} = \int_{\mathbb{R}^d} h_1(x) h_2(x)\, dx.$$

This equality follows from density of $\mathscr{Z}$ in $\dot{B}^a_b \cap L^2$ for any $a \in \mathbb{R}$ and $1 \leqslant b < \infty$ which can be proved by the same procedure as was used in the proof of Theorem 5.1.5 in [30], cf. Theorem 2.3.3 in [30].

Lemma A.5. *Let $r, a, b \in [1, \infty]$ and $s \in \mathbb{R}$ satisfy*

$$\frac{1}{a} + \frac{1}{b} = 1 + \frac{1}{r}.$$

Then

$$\|S * \varphi\|_{\dot{B}^s_r} \leqslant \min\{\|S\|_{\dot{B}^s_a} \|\varphi\|_{L^b(\mathbb{R}^d)}, \|\varphi\|_{\dot{B}^s_a} \|S\|_{L^b(\mathbb{R}^d)}\}$$

holds for any tempered distribution $S \in \mathscr{S}'_{\mathbb{C}}$ and for any $\varphi \in \mathscr{S}_{\mathbb{C}}$. If φ is a smooth compactly supported density on $\mathbb{R}^d$, $\varphi_n(x) = n^d \varphi(nx)$ and $1 \leqslant r < \infty$ then

$$\lim_{n\to\infty} \|S * \varphi_n - S\|_{\dot{B}^s_r} = 0, \quad S \in \mathscr{S}'_{\mathbb{C}} \cap \dot{B}^s_r.$$

Proof. The inequality

$$\left\|(S*\varphi)*\mathscr{F}_{-1}\phi_j\right\|_{L^r}=\|S*\mathscr{F}_{-1}\phi_j*\varphi\|_{L^r}\leqslant\min\left\{\|S*\mathscr{F}_{-1}\phi_j\|_{L^a}\|\varphi\|_{L^b},\|\varphi*\mathscr{F}_{-1}\phi_j\|_{L^a}\|S\|_{L^b}\right\}$$

holds by the Young inequality. In the second case,

$$\left\|(S*\varphi_n)*\mathscr{F}_{-1}\phi_j-S*\mathscr{F}_{-1}\phi_j\right\|_{L^r}=\left\|(S*\mathscr{F}_{-1}\phi_j)*\varphi_n-S*\mathscr{F}_{-1}\phi_j\right\|_{L^r}$$

converges to zero as $n\to\infty$ and is bounded by $2\|S*\mathscr{F}_{-1}\phi_j\|_{L^r}$. Hence the second claim follows from the Lebesgue dominated convergence theorem. □

A.2. Measurability in homogeneous Besov spaces

Proposition A.6. *Let $1\leqslant r<\infty$, $s\in\mathbb{R}$, let $(X,\mathcal{X})$ be a measure space and let $S:X\to\mathscr{Z}'$ be such that $\langle S,\varphi\rangle$ is $\mathcal{X}$-measurable for every $\varphi\in\mathscr{Z}$. Then $A=\{x\in X\colon S(x)\in\dot{B}^s_r\}\in\mathcal{X}$ and $\mathbf{1}_AS:\Omega\to\dot{B}^s_r$ is Borel measurable.*

Proof. Let $T\in\dot{B}^s_r$. The mapping $X\times\mathbb{R}^d\to\mathbb{C}:(x,z)\mapsto(S_x-T)*\mathscr{F}^{-1}\phi_j(z)$ is $\mathcal{X}$-measurable in the first variable and continuous in the second one, hence, by the Carathéodory theorem, it is jointly measurable. Consequently, $x\mapsto\|(S_x-T)*\mathscr{F}^{-1}\phi_j\|_{L^r}$ is $\mathcal{X}$-measurable and so $x\mapsto\|S_x-T\|_{\dot{B}^s_r}$ is $\mathcal{X}$-measurable. In particular, $A\in\mathcal{X}$. The sets $\{x\colon \mathbf{1}_AS_x\in K\}$ belong to $\mathcal{X}$ if K is a ball in $\dot{B}^s_r$ and, since $\dot{B}^s_r$ is separable by Proposition A.1, $\mathbf{1}_AS:X\to\dot{B}^s_r$ is Borel measurable. □

A.3. Transformations in homogeneous Besov and Lebesgue spaces

We use the classical notation

$$\alpha(r)=\frac{1}{2}-\frac{1}{r},\qquad\beta(r)=\frac{d+1}{2}\alpha(r),\qquad\gamma(r)=(d-1)\alpha(r),\qquad\delta(r)=d\alpha(r)$$

and the convention $\|u\|_{L^t}=(\int_{\mathbb{R}^d}|u|^t\,dx)^{\frac{1}{t}}$ for $t>0$.

Lemma A.7. *Let $q=2\frac{d+1}{d-1}$, $s\in\mathbb{R}$ such that $1\leqslant\delta(s)\leqslant\frac{3d+1}{2(d+1)}$, $a\in(0,\infty)$, $r\in[2,\infty)$, $p\in(0,\frac{1}{2}]$, $b=\frac{ad}{\frac{1}{2}+\delta(r)+\delta(q)-p}$, $\theta=q(\delta(s)-1)$ and $h:\mathbb{R}\to\mathbb{R}$ be a locally Lipschitz function satisfying $h(0)=0$ and $|h'(t)|\leqslant c|t|^a$ for a.e. $t\in\mathbb{R}$. Then there exists $C<\infty$ such that*

$$\|u\|_{L^s}\leqslant C\|u\|^{\theta}_{\dot{B}^{\frac{1}{2}}_q}\|u\|^{1-\theta}_{L^{\frac{2d}{d-2}}},\qquad\left\|h(u)\right\|_{\dot{B}^p_{r'}}\leqslant C\|u\|_{\dot{B}^{\frac{1}{2}}_q}\|u\|^a_{L^b}$$

hold for every $u\in H^1(\mathbb{R}^d)$. If $a=0$ then $\|h(u)\|_{\dot{B}^p_2}\leqslant c\|u\|_{H^1}$ holds for every $u\in H^1(\mathbb{R}^d)$.

Proof. Let $a>0$. Then Lemma 3.2 in [11] implies that

$$\left\|h(u)\right\|_{\dot{B}^p_{r'}}\leqslant c\|u\|_{\dot{B}^p_t}\|u\|^a_{L^b}$$

holds for every $u\in H^1(\mathbb{R}^d)$ where $t\in[q,\infty)$ is chosen such that $p+\delta(t)=\frac{1}{2}+\delta(q)$, and $\|u\|_{\dot{B}^p_t}\leqslant\|u\|_{\dot{B}^{\frac{1}{2}}_q}$ by the Sobolev embedding (A.12) in [11]. It holds that $\dot{B}^{\frac{1}{2}}_q\subseteq\dot{B}^0_c\subseteq L^c$ where $c=\frac{2d(d+1)}{d^2-2d-1}$ by the Sobolev embedding (e.g. Lemma A.2 in [11]), the generalized Minkowski inequality and Theorem 5.2.3/1 in [30]. Hence, by the Riesz–Thorin interpolation,

$$\|u\|_{L^s} \leqslant \|u\|_{L^c}^{\theta} \|u\|_{L^{\frac{2d}{d-2}}}^{1-\theta} \leqslant c_1 \|u\|_{\dot{B}_q^{\frac{1}{2}}}^{\theta} \|u\|_{L^{\frac{2d}{d-2}}}^{1-\theta}.$$

If $a = 0$ then Lemma 3.2 in [11] implies that $\|h(u)\|_{\dot{B}_2^p} \leqslant c\|u\|_{\dot{B}_2^p}$ and

$$\|u\|_{\dot{B}_2^p} \leqslant c_0 \|u\|_{\dot{B}_2^0}^{1-p} \|u\|_{\dot{B}_2^1}^{p} \leqslant c_1 \|u\|_{L^2}^{1-p} \|u\|_{\dot{H}^1}^{p} \leqslant c\|u\|_{H^1}$$

by interpolation and the fact that $\dot{B}_2^0 = L^2$ and $\dot{B}_2^1 = \dot{H}^1$ e.g. by Theorem 5.2.3/1 and Remark 5.2.3/1 in [30]. □

Lemma A.8. *Let $q = 2\frac{d+1}{d-1}$, $0 \leqslant w \leqslant \frac{d-1}{d+1}$, $a \in [0, \infty)$, $h : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function such that $h(0) = 0$ and $|h'(t)| \leqslant c|t|^a$ for a.e. $t \in \mathbb{R}$. There exists a constant C such that*

$$\bigl\|K_t\bigl(e^{i\langle y,\cdot\rangle} h(u)\bigr)\bigr\|_{\dot{B}_q^{\frac{1}{2}}} + \bigl\|\dot{K}_t\bigl(e^{i\langle y,\cdot\rangle} h(u)\bigr)\bigr\|_{\dot{B}_q^{-\frac{1}{2}}} \leqslant C\bigl(\|u\|_{H^1} + |y|^{\frac{1}{q}} \|u\|_{L^2}\bigr)$$

holds for every $u \in H^1$, $t \in \mathbb{R}$ and $y \in \mathbb{R}^d$ if $a = 0$ (the case $w = 0$),

$$\bigl\|K_t\bigl(h(u)\bigr)\bigr\|_{\dot{B}_q^{\frac{1}{2}}} + \bigl\|\dot{K}_t\bigl(h(u)\bigr)\bigr\|_{\dot{B}_q^{-\frac{1}{2}}} \leqslant C|t|^{-\frac{d-1}{d+1}} \|u\|_{\dot{B}_q^{\frac{1}{2}}}^{q-1} \|u\|_{L^{\frac{2d}{d-2}}}^{\frac{4}{(d-2)(d-1)}}$$

holds for every $u \in H^1$ and $t \in \mathbb{R}$ if $a = \frac{4}{d-2}$ (the case $w = \frac{d-1}{d+1}$ and $y = 0$),

$$\begin{aligned}\bigl\|K_t\bigl(e^{i\langle y,\cdot\rangle} h(u)\bigr)\bigr\|_{\dot{B}_q^{\frac{1}{2}}} + \bigl\|\dot{K}_t\bigl(e^{i\langle y,\cdot\rangle} h(u)\bigr)\bigr\|_{\dot{B}_q^{-\frac{1}{2}}} &\leqslant C|t|^{-w} \|u\|_{\dot{B}_q^{\frac{1}{2}}}^{1+q(\frac{1}{2}-w)} \|u\|_{L^{\frac{2d}{d-2}}}^{\frac{3}{d-2}-q(\frac{1}{2}-w)} \\ &\quad + C|t|^{-w} |y|^{\frac{1}{q}+\frac{w}{d-1}} \|u\|_{\dot{B}_q^{\frac{1}{2}}}^{q(\frac{1}{2}-\frac{dw}{d-1})} \|u\|_{L^{\frac{2d}{d-2}}}^{\frac{d+1}{d-2}-q(\frac{1}{2}-\frac{dw}{d-1})}\end{aligned}$$

holds for every $u \in H^1$, $t \in \mathbb{R}$ if $a = \frac{3}{d-2}$, $\frac{1}{q} \leqslant w \leqslant \frac{d-1}{2d}$ and $y \in \mathbb{R}^d$, and

$$\bigl\|K_t\bigl(e^{i\langle y,\cdot\rangle} h(u)\bigr)\bigr\|_{\dot{B}_q^{\frac{1}{2}}} + \bigl\|\dot{K}_t\bigl(e^{i\langle y,\cdot\rangle} h(u)\bigr)\bigr\|_{\dot{B}_q^{-\frac{1}{2}}} \leqslant C\bigl(\|u\|_{\dot{B}_q^{\frac{1}{2}}} \|u\|_{L^{\frac{2d}{d-2}}}^{\frac{2}{d-2}} + |y|^{\frac{1}{q}} \|u\|_{L^{\frac{2d}{d-2}}}^{\frac{d}{d-2}}\bigr)$$

holds for every $y \in \mathbb{R}^d$, $u \in H^1$ and $t \in \mathbb{R}$ if $a = \frac{2}{d-2}$ (the case $w = 0$).

Proof. Let us define $I_\sigma \xi = \mathscr{F}_{-1}\{|x|^\sigma \cdot \mathscr{F}\xi\}$ and $U_\sigma \xi = \mathscr{F}_{-1}\{e^{it|x|} \cdot \mathscr{F}\xi\}$ for $\sigma \in \mathbb{R}$ and $\xi \in \mathscr{Z}'$. Then, for $r \in \mathbb{R}$ such that $w = \gamma(r)$,

$$\begin{aligned}\|U_t\xi\|_{\dot{B}_q^{-\frac{1}{2}}} &\leqslant c_0 \|U_t\xi\|_{\dot{B}_r^s} \leqslant c_1 \bigl\|U_t(I_{s+\beta(r)}\xi)\bigr\|_{\dot{B}_r^{-\beta(r)}} \leqslant c_3 |t|^{-w} \|I_{s+\beta(r)}\xi\|_{\dot{B}_{r'}^{\beta(r)}} \\ &\leqslant c_4 |t|^{-w} \|\xi\|_{\dot{B}_{r'}^p}, \quad t \neq 0,\ \xi \in \mathscr{Z}',\end{aligned}$$

holds by (A.12) in [11], Theorem 5.2.3/1(i) in [30] and the inequality (3.14) in [10] where $s = \delta(q) - \delta(r) - \frac{1}{2} = \frac{1}{q} - \frac{dw}{d-1}$ and $p = s + 2\beta(r) = \frac{1}{q} + \frac{w}{d-1} \in [\frac{1}{q}, \frac{1}{2}]$. Applying Lemma 3.2 in [11] and Lemma A.7, we get

$$\begin{aligned}\left\|e^{i\langle y,\cdot\rangle}h(u)\right\|_{\dot B^p_{r'}} &\leqslant c_5\left(\left\|h(u)\right\|_{\dot B^p_{r'}}+|y|^p\left\|h(u)\right\|_{L^{r'}}\right)\\ &\leqslant c_6\left(\|u\|_{\dot B^{\frac12}_q}\|u\|^a_{L^{\frac{ad}{1+w}}}+|y|^p\|u\|^{a+1}_{L^{(a+1)r'}}\right)\end{aligned}$$

if $a>0$ and the result now follows from the Riesz–Thorin interpolation part of Lemma A.7. If $a=w=0$ then $r=2$ and so

$$\begin{aligned}\left\|e^{i\langle y,\cdot\rangle}h(u)\right\|_{\dot B^p_2} &\leqslant c_5\left(\left\|h(u)\right\|_{\dot B^p_2}+|y|^p\left\|h(u)\right\|_{L^2}\right)\\ &\leqslant c_6\left(\|u\|_{H^1}+|y|^p\|u\|_{L^2}\right)\end{aligned}$$

by Lemma A.7. □

Lemma A.9. *Let $q=2\frac{d+1}{d-1}$, $s=2\frac{d+1}{d-2}$ and let $h_i:\mathbb{R}\to\mathbb{R}$, $i=1,2$, be locally Lipschitz functions satisfying $h_i(0)=0$, $|h_1'(t)|\leqslant c(1+|t|^{\frac{4}{d-2}})$ and $|h_2'(t)|\leqslant c(1+|t|^{\frac{3}{d-2}})$ for a.e. $t\in\mathbb{R}$. Then*

$$\begin{aligned}\left\|K_t\big(h_1(\varphi_1)-h_1(\varphi_2)\big)\right\|_{L^q} &\leqslant c\big(|t|+|t|^{-\frac{d+1}{d-1}}\big)\|\varphi_1-\varphi_2\|_{L^q}\big(1+\|\varphi_1\|_{L^s}^{\frac{4}{d-2}}+\|\varphi_2\|_{L^s}^{\frac{4}{d-2}}\big),\\ \left\|K_t\big(e^{i\langle y,\cdot\rangle}\big[h_2(\varphi_1)-h_2(\varphi_2)\big]\big)\right\|_{L^q} &\leqslant c\big(|t|+|t|^{-\frac{1}{s}}\big)\|\varphi_1-\varphi_2\|_{L^q}\big(1+\|\varphi_1\|_{L^s}^{\frac{3}{d-2}}+\|\varphi_2\|_{L^s}^{\frac{3}{d-2}}\big)\end{aligned}$$

hold for every $t\neq0$, $\varphi_i\in H^1$ and $y\in\mathbb{R}^d$ where K_t was defined in (9.1)*.*

Proof. See (3.44) and (3.45) in [11] for the first inequality and Lemma 27 in [22] for the second one. □

Proposition A.10. *Let $q=2\frac{d+1}{d-1}$ and consider the operators (T_t) introduced in* (9.1)*. Then there exists a constant $C<\infty$ such that*

$$\int_{-\infty}^{\infty}\|T_tz\|^q_{\dot B^{\frac12}_q\oplus\dot B^{-\frac12}_q}\,dt\leqslant C\|z\|^q_{H^1\oplus L^2}$$

holds for every $z\in H^1\oplus L^2$.

Proof. See Proposition 3.1 in [10]. □

Proposition A.11. *Let $2\leqslant r<\infty$, $s\in\mathbb{R}$ and let $X\in\{L^r,\dot B^s_r\}$. Then there exists a constant $c_r<\infty$ such that*

$$\|O\|^2_{\mathscr{L}_2(H_\mu,X)}\leqslant c_r(2\pi)^{-d}\int_{\mathbb{R}^d}\left\|O\left(e^{i\langle y,\cdot\rangle}\right)\right\|^2_X\mu(dy)$$

holds, provided that the right-hand side is finite, for any operator $O:C_b\to\mathscr{Z}'$ defined as

$$O\xi=\mathscr{F}_{-1}\big\{m\cdot\mathscr{F}(h\cdot\xi)\big\},\quad \xi\in C_b,$$

where h is a real tempered function on $\mathbb{R}^d$ and $m\in C^\infty(\mathbb{R}^d\setminus\{0\})$ satisfies

$$\left|D^\alpha m(x)\right|\leqslant c_\alpha\left(|x|^{-n_\alpha}+|x|^{n_\alpha}\right),\quad x\neq0,$$

for every multiindex α and some $n_\alpha\in\mathbb{N}$.

Proof. See Theorem 28 in [22]. □

Lemma A.12 *(Hardy–Littlewood–Sobolev). Let $a \in (0,1)$ and $r \in (\frac{1}{a}, \infty)$. Then there exists $c < \infty$ such that*

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |t-s|^{-a} \varphi(s)\psi(s)\,ds \right)^r dt \leqslant c\|\varphi\|^r_{L^r(\mathbb{R})} \|\psi\|^r_{L^{\frac{1}{1-a}}(\mathbb{R})}$$

holds for every measurable nonnegative functions φ, ψ.

Proof. It holds that $0 < 1 + \frac{1}{r} - a < 1$ so there exists $1 < p < \infty$ such that $a + \frac{1}{p} = 1 + \frac{1}{r}$. Hence, by the Hardy–Littlewood–Sobolev theorem, we get

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |t-s|^{-a} \varphi(s)\psi(s)\,ds \right)^r dt \leqslant c\|\varphi\psi\|^r_{L^p(\mathbb{R})} \leqslant c\|\varphi\|^r_{L^r(\mathbb{R})} \|\psi\|^r_{L^{\frac{1}{1-a}}(\mathbb{R})}. \qquad \square$$

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