An approach to identifying linearizable dynamic errors-in-variables systems

Levente Hunyadi and István Vajk

***Abstract*— We consider an important class of dynamic single-input single-output nonlinear systems where the system model is polynomial in observations but linear in parameters. The investigation is done in the errors-in-variables framework, i.e. both input and output are observed with noise. Assuming white Gaussian measurement noise that is characterized by a magnitude and a covariance structure, we propose a nonlinear extension to the generalized Koopmans–Levin method that can estimate parameters of dynamic nonlinear systems with polynomial nonlinearities given a priori knowledge on the noise covariance structure. In order to estimate noise structure, we apply a covariance matching objective function. Combining the extended Koopmans–Levin and the covariance matching approaches, an identification algorithm to estimate both model and noise parameters is proposed. The feasibility of the approach is demonstrated by Monte-Carlo simulations.**

***Keywords*—dynamic system identification; linearizable nonlinear systems; polynomial eigenvalue problem; covariance matching**

I. Introduction

Errors-in-variables systems where all variables are equally observed with noise are of particular significance in applications where the quantitative description of the internal laws constituting the system is of interest rather than predicting future behavior. Such applications include computer vision, image reconstruction, speech and audio processing, signal processing, modal and spectral analysis, system identification, econometrics and time series analysis. In these applications, the task is to construct a best possible system model based on noisy observations. A general system model takes the form

$$f\left(\theta,\, z_{0,i}\right) = 0$$

where the vector $z_{0,i}$ denotes (noise-free) data for $\forall i = 1, \ldots, N$, N being the number of observations, the vector θ encapsulates the parameters of interest, and f represents some constraint between (for dynamic systems, past and present) data. A usual assumption that is satisfied in most applications is that the constraint f is linear in θ, i.e.

$$\theta^\top g\left(z_{0,i}\right) = 0 \tag{1}$$

where g is a linearization of f. Given that $z_{0,i}$ are not directly observable but contaminated with noise, the most common assumption being Gaussian white noise, hence the actual observations z_i satisfy $z_i = z_{0,i} + \tilde{z}_i$, the objective is to derive estimates for θ given the noisy observations z_i.

The authors are with the Department of Automation and Applied Informatics, Budapest University of Technology and Economics, H-1111 Budapest, Goldmann György tér 3., Hungary; Phone: +36 1 463-2870; Fax: +36 1 463-2871; Email: {hunyadi,vajk}@aut.bme.hu

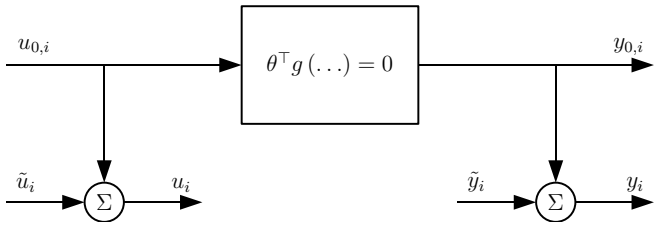

Fig. 1. The basic setup for a linearizable discrete-time dynamic errors-in-variables system. The function g relates current output and a finite number of past inputs and outputs.

For the special case of identifying linear errors-in-variables systems, where the constraint f is linear not only in θ but also in data $z_{0,i}$, a number of estimation schemes have been proposed. For dynamic single-input single-output (SISO) systems, where the system is described by the linear equation

$$\begin{aligned} y_{0,i} \quad &+ \quad a_1 y_{0,i-1} + a_2 y_{0,i-2} + \ldots + a_m y_{0,i-m} = \\ &= \quad b_1 u_{0,i-1} + b_2 u_{0,i-2} + \ldots + b_n u_{0,k-m} \end{aligned}$$

or more compactly,

$$A(q^{-1}) y_{0,i} = B(q^{-1}) u_{0,i}$$

where the polynomial variable q^{-1} denotes the backward shift operator $\circ_i q^{-1} = \circ_{i-1}$ (in which $\circ$ is a generic placeholder for a parameter) but neither the noise-free true input $u_{0,i}$ nor the true output $y_{0,i}$ is observable but one is confined to their noise-contaminated variants u and y, proposed methods include bias-compensating least squares [3], the Frisch scheme [2], instrumental variable [9], higher-order statistics [10], structured total least squares [7], frequency-domain and efficient maximum likelihood methods, see [8] for a comprehensive survey.

This paper deals with a nonlinear extension of the generalized Koopmans–Levin method to estimate model parameters of a dynamic system with given noise structure where the linearization g in (1) is a polynomial in terms of input and output data (Figure 1), and a subsequent covariance matching objective function to estimate noise covariance structure. The Koopmans–Levin method, proposed in [6], gives a non-iterative quick estimate of the model parameters of a linear system given a priori information on the noise structure. The original method was generalized in [11] to improve estimation accuracy at the cost of increased computational complexity, incorporating as special cases the original Koopmans–Levin method and the maximum likelihood method. In addition, a nonlinear extension to the

original Koopmans method was proposed in [12] for static systems.

The Koopmans–Levin method and its generalization are briefly described in Section II. Section III combines and extends the results of [11] and [12] to nonlinear dynamic systems that comprise of polynomial nonlinearities yet are linear in model parameters. The outlined method assumes a preliminarily known noise structure, Section IV extends the estimation method so that no such assumptions are required. In order to demonstrate the feasibility of the method, some simulation results are presented in Section V before the paper concludes with Section VI.

II. The generalized Koopmans–Levin method

Consider the linear SISO errors-in-variables system $G(q^{-1})$ described by the autoregressive moving average (ARMA) difference equation

$$A(q^{-1})y_{0,i} = B(q^{-1})u_{0,i} \tag{2}$$

with

$$\begin{aligned} A(q^{-1}) &= a_0 + a_1 q^{-1} + \cdots + a_m q^{-m} \\ B(q^{-1}) &= b_0 + b_1 q^{-1} + \cdots + b_m q^{-m}. \end{aligned}$$

Given the aforementioned system description, we may introduce the model parameter vector θ and the extended regressor vector z

$$\begin{aligned} \theta^\top &= \begin{bmatrix} a_0 & a_1 & \dots & a_m & -b_0 & -b_1 & \dots & -b_m \end{bmatrix} \\ z_i^\top &= \begin{bmatrix} y_i & \dots & y_{i-m} & u_i & \dots & u_{i-m} \end{bmatrix} \end{aligned}$$

such that $\theta^\top z_{0,i} = 0 \quad \forall i = m+1 \dots N$ with m being the order of the model and $\tilde{z}_i = z_i - z_{0,i}$ is the noise contribution following a normal distribution such that $\tilde{y}_i \sim N(0,\, \sigma_y^2)$ and $\tilde{u}_i \sim N(0,\, \sigma_u^2)$. Notice that the system is linear in components $y_{0,i}$ and $u_{0,i}$ as well as in model parameters a_k and b_k with $0 \leqq k \leqq m$. Furthermore, introduce the observation sample and noise covariance matrices as

$$\begin{aligned} D &= \mathbb{E}\left(zz^\top\right) \approx \frac{1}{N}\sum_{i=1}^{N} z_i z_i^\top \\ \mu C &= \mathbb{E}\left(\tilde{z}\tilde{z}^\top\right) \approx \frac{1}{N}\sum_{i=1}^{N} \tilde{z}_i \tilde{z}_i^\top \end{aligned}$$

with μ denoting noise magnitude and C representing a normalized noise covariance matrix, or noise (covariance) structure, which is a diagonal matrix due to our assumptions on noise. We assume that the input signal u_0 is an excitation of sufficient order to make identification possible.

The essence of the Koopmans–Levin method is that the (full-rank) sample covariance matrix D comprising of (noisy) observations can be decomposed into a (rank-deficient) noise-free component D_0 and a noise component C:

$$\theta^\top D\theta = \theta^\top D_0\theta + \theta^\top \mu C\theta = \theta^\top \mu C\theta$$

in which $\theta^\top D\theta = \theta^\top \mathbb{E}\left(zz^\top\right)\theta = \mathbb{E}\left(\theta^\top zz^\top\theta\right)$ and $\theta^\top D_0\theta = \theta^\top \mathbb{E}\left(z_0 z_0^\top\right)\theta = 0$ so that finding θ entails minimizing the objective function

$$J = \frac{1}{2}\frac{\theta^\top D\theta}{\theta^\top C\theta} \tag{3}$$

with $\|\theta\| = 1$, which can be effectively tackled by solving the eigenvector decomposition problem

$$(D - \mu C)\,\theta = 0$$

or

$$\det(D_0) = \det(D - \mu C) = 0$$

so that the model parameter vector is found by solving a generalized eigenvector problem on the matrix pair $(D,\, C)$. The problem may alternatively be formulated using matrix notation where

$$\begin{aligned} Z &= \begin{bmatrix} y_1 & \cdots & y_m & u_1 & \cdots & u_m \\ y_2 & \cdots & y_{m+1} & u_2 & \cdots & u_{m+1} \\ \vdots & & \vdots & \vdots & & \vdots \end{bmatrix} \\ D &= Z^\top Z \\ \mu C &= \operatorname{diag}\begin{bmatrix} \sigma_y^2 & \cdots & \sigma_y^2 & \sigma_u^2 & \cdots & \sigma_u^2 \end{bmatrix} \end{aligned}$$

with Z being an $(N - m + 1) \times 2m$ matrix and $C = \mu C_\rho \otimes I_m$ denoting the noise structure such that the noise covariance matrix is known up to a multiplication by a scalar μ representing the noise magnitude, i.e.

$$\mu C_\rho = \mu \begin{bmatrix} \sin^2\rho & 0 \\ 0 & \cos^2\rho \end{bmatrix} = \begin{bmatrix} \sigma_y^2 & 0 \\ 0 & \sigma_u^2 \end{bmatrix} \tag{4}$$

in which we assume that the noise structure matrix C_ρ (uniquely determined by the noise direction ρ that reflects the relative distribution of input vs. output noise) is preliminarily known.

The variance of the estimates thus obtained, however, is rather large. One way [11] to improve the robustness of the parameter estimation approach outlined above is by instead of (3) minimizing the objective function

$$J = \frac{1}{2}\frac{1}{q-m}\operatorname{trace}\left(G_q^\top Z_q^\top \left(G_q^\top C_q G_q\right)^{-1} Z_q G_q\right)$$

where Z_q is an $(N - q + 1) \times 2q$ matrix obtained by augmenting Z with $q-m$ columns of additional observations for both y and u; G_q is a $2q \times (q - m)$ matrix of model parameters such that $Z_{0,q}G_q = 0$; and $C_q = (\mu C_\rho) \otimes I_q$ is a diagonal covariance structure matrix of size $2q$, and $m + 1 \leqq q \leqq N$; i.e.

$$G_\circ = \begin{bmatrix} \circ_0 & & & \\ \circ_1 & \circ_0 & & \\ \vdots & \circ_1 & & \\ \circ_m & \vdots & \ddots & \circ_0 \\ & \circ_m & & \circ_1 \\ & & \ddots & \vdots \\ & & & \circ_m \end{bmatrix}_{q,\, q-m}$$

$$G_q = \begin{bmatrix} G_y \\ -G_u \end{bmatrix}_{2q,\, q-m}$$

$$Z_q = \begin{bmatrix} y_1 & \cdots & y_q & u_1 & \cdots & u_q \\ y_2 & \cdots & y_{q+1} & u_2 & \cdots & u_{q+1} \\ \vdots & & \vdots & \vdots & & \vdots \\ y_{N-q+1} & \cdots & y_N & u_{N-q+1} & \cdots & u_N \end{bmatrix}.$$

In short, both the model parameter vector θ and the original observation matrix Z have been extended from size m to q.

The above problem can be reformulated as

$$J = \frac{1}{2(q-m)} \text{trace}\left(\left(G_q^\top C_q G_q\right)^{-1} G_q^\top D_q G_q\right) \quad (5)$$

which can be gradually approximated with the iteration scheme

$$\frac{\text{trace}\left(\left(G_q^\top(\theta_k) C_q G_q(\theta_k)\right)^{-1} G_q^\top(\theta) D_q G_q(\theta)\right)}{\text{trace}\left(\left(G_q^\top(\theta_k) C_q G_q(\theta_k)\right)^{-1} G_q^\top(\theta) C_q G_q(\theta)\right)} \quad (6)$$

where $\theta_{k+1} = \arg\min_\theta(6)$ and k stands for the iteration count, which yields the same extreme value upon convergence. To facilitate easier computation, the above scheme is equivalent to

$$\theta_{k+1} = \arg\min_\theta \frac{\theta^\top T^\top \left(\left(G_q^\top(\theta_k) C_q G_q(\theta_k)\right)^{-1} \otimes D_q\right) T\theta}{\theta^\top T^\top \left(\left(G_q^\top(\theta_k) C_q G_q(\theta_k)\right)^{-1} \otimes C_q\right) T\theta} \quad (7)$$

where T (a sparse matrix of zeros and ones) is chosen such that $\text{vec}(G_q) = T\theta$. In each iteration, minimization w.r.t. θ is attained by solving a generalized eigenvector decomposition problem on the matrix pair $(Q,\, R)$ with

$$\begin{aligned} Q &= T^\top \left(\left(G_q^\top(\theta_k) C_q G_q(\theta_k)\right)^{-1} \otimes D_q\right) T \\ R &= T^\top \left(\left(G_q^\top(\theta_k) C_q G_q(\theta_k)\right)^{-1} \otimes C_q\right) T \end{aligned}$$

where θ is the eigenvector that belongs to the smallest eigenvalue μ.

III. A NONLINEAR EXTENSION

In order to further generalize the Koopmans–Levin method to nonlinear systems, linear components y_i and u_i give way to the nonlinearities that occur in the model. For instance, for the components y_i, u_i, u_i^2 and $y_i u_i$, the $(N-q+1)\times nq$ matrix Z_q takes the form

$$Z_q = \begin{bmatrix} y_1 & \cdots & y_q & u_1 & \cdots & u_q & u_1^2 & \cdots & u_q^2 & y_1u_1 & \cdots & y_qu_q \\ y_2 & \cdots & y_{q+1} & u_2 & \cdots & u_{q+1} & u_2^2 & \cdots & u_{q+1}^2 & y_2u_2 & \cdots & y_{q+1}u_{q+1} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \end{bmatrix}$$

and the general form for the matrix G_q becomes

$$G_q = \begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_n \end{bmatrix}_{nq,\, q-m}$$

n being the number of nonlinear components (in our case, $n = 4$) and G_k encapsulating the parameters for the respective nonlinearity, $0 < k \leqq n$. Notice that the matrix product $Z_q G_q$ entails that the system is still linear in parameters. However, the covariance matrix structure C_q is no longer a single diagonal matrix but is replaced by a matrix polynomial

$$C_q(\mu) = \mu C_q^{(1)} + \mu^2 C_q^{(2)} + \ldots + \mu^p C_q^{(p)}$$

in which $C_q^{(k)}$ is the kth coefficient of the matrix polynomial $C_q(\mu)$. One can use the following identities in deriving $C_q(\mu)$:

$$\begin{aligned} \mathbb{E}\left(x_i^p\right) &= \mathbb{E}\left(x_{0,i} + n_i\right)^p \\ \mathbb{E}\left(n_i^{2p}\right) &= (2p-1)(2p-3)\ \ldots\ 1\,\sigma^{2p} \\ \mathbb{E}\left(n_i^{2p-1}\right) &= 0 \\ \mathbb{E}\left(x_i\right) &\approx \bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i \\ \mathbb{E}\left(x_i x_{i-\tau}\right) &= \mathbb{E}\left(x_i\right)\mathbb{E}\left(x_{i-\tau}\right) \end{aligned}$$

in which n_i is a zero-mean σ^2-variance normally distributed random value at time instant i. [12] features a detailed description as well as an introductory example on how to derive these terms. Figure 2 shows a sample covariance polynomial for $q = 2$. Observe that the matrix entries usually depend not only on noise parameters σ_u and σ_y but also on (means of) the observations themselves.

The use of covariance matrix polynomials instead of regular covariance matrices necessitates some modifications to the objective function (5) as well as the iteration scheme (7). By including the noise magnitude within the trace operator in (5) yielding

$$\text{trace}\left(\left(G_q^\top \mu C_q G_q\right)^{-1} G_q^\top D_q G_q\right),$$

it is apparent that the matrix product approaches the unit matrix should the best possible model parameters and noise covariance matrix be used. In this spirit, (5) can be reformulated as

$$J = \left(\text{trace}\left(\left(G_q^\top \mu C_q G_q\right)^{-1} G_q^\top D_q G_q\right) - (q-m)\right)^2 \quad (8)$$

with m being the the order of the dynamic model to estimate where the minimum of J is attained when both model and noise magnitude estimates best match observations. Substituting the noise covariance polynomial $C_q(\mu)$ into (8), we get a parameter estimation scheme for nonlinear systems. Thus, we propose the following differentiable objective function:

$$J = \left(\text{trace}\left(\gamma^{-1}\delta\right) - (q-m)\right)^2 \quad (9)$$

where

$$\begin{aligned} \delta &= G_q^\top(\theta) D_q G_q(\theta) \\ \gamma &= G_q^\top(\theta) C_q(\mu) G_q(\theta) \end{aligned}$$

As the function (9) is differentiable, a direct search utilizing the Levenberg-Marquardt method yields model parameter

$$C(\mu) = \mu \begin{bmatrix} \sigma_y^2 & 0 & 0 & 0 & \bar{y}\sigma_u^2 & \bar{y}\sigma_u^2 & \bar{u}\sigma_y^2 & 0 \\ 0 & \sigma_y^2 & 0 & 0 & \bar{y}\sigma_u^2 & \bar{y}\sigma_u^2 & 0 & \bar{u}\sigma_y^2 \\ 0 & 0 & \sigma_u^2 & 0 & 3\bar{u}\sigma_u^2 & \bar{u}\sigma_u^2 & \bar{y}\sigma_u^2 & 0 \\ 0 & 0 & 0 & \sigma_u^2 & \bar{u}\sigma_u^2 & 3\bar{u}\sigma_u^2 & 0 & \bar{y}\sigma_u^2 \\ \bar{y}\sigma_u^2 & \bar{y}\sigma_u^2 & 3\bar{u}\sigma_u^2 & \bar{u}\sigma_u^2 & 6\sigma_u^2\bar{u}^2 & 2\sigma_u^2\bar{u}^2 & 3\bar{y}\bar{u}\sigma_u^2 & \bar{y}\bar{u}\sigma_u^2 \\ \bar{y}\sigma_u^2 & \bar{y}\sigma_u^2 & \bar{u}\sigma_u^2 & 3\bar{u}\sigma_u^2 & 2\sigma_u^2\bar{u}^2 & 6\sigma_u^2\bar{u}^2 & \bar{y}\bar{u}\sigma_u^2 & 3\bar{y}\bar{u}\sigma_u^2 \\ \bar{u}\sigma_y^2 & 0 & \bar{y}\sigma_u^2 & 0 & 3\bar{y}\bar{u}\sigma_u^2 & \bar{y}\bar{u}\sigma_u^2 & \bar{y}^2\sigma_u^2 + \sigma_y^2\bar{u}^2 & 0 \\ 0 & \bar{u}\sigma_y^2 & 0 & \bar{y}\sigma_u^2 & \bar{y}\bar{u}\sigma_u^2 & 3\bar{y}\bar{u}\sigma_u^2 & 0 & \bar{y}^2\sigma_u^2 + \sigma_y^2\bar{u}^2 \end{bmatrix} - \mu^2 \begin{bmatrix} \cdots & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & 3\sigma_u^4 & \sigma_u^4 & 0 & 0 \\ & \sigma_u^4 & 3\sigma_u^4 & 0 & 0 \\ & 0 & 0 & \sigma_y^2\sigma_u^2 & 0 \\ \cdots & 0 & 0 & 0 & \sigma_y^2\sigma_u^2 \end{bmatrix}$$

Fig. 2. Example covariance matrix for the polynomial components y_i, u_i, u_i^2 and $y_i u_i$ with $q = 2$. Entries not indicated take a value of zero.

and noise magnitude estimates. However, the Levenberg-Marquardt method finds a single local minimum in a single run, making the scheme sensitive to initial values.

Iterative schemes based on matrix decomposition are more robust against getting stuck in local minima as they deliver multiple local minima in a single iteration. Modifying (7) to incorporate the covariance polynomial $C_q(\mu)$ we get

$$\bar{\theta}_{k+1} = \arg\min_{\bar{\theta}} \frac{\theta^\top T^\top \left(\left(G_q^\top(\theta_k) C_q(\mu_k) G_q(\theta_k) \right)^{-1} \otimes D_q \right) T\theta}{\theta^\top T^\top \left(\left(G_q^\top(\theta_k) C_q(\mu_k) G_q(\theta_k) \right)^{-1} \otimes C_q(\mu) \right) T\theta} \tag{10}$$

with $\bar{\theta} = \begin{bmatrix} \theta & \mu \end{bmatrix}$ and $\|\theta\| = 1$ needs the solution of a polynomial eigenvalue decomposition problem

$$\Psi(\mu)\theta = \left(Q - \mu R_1 - \mu^2 R_2 - \ldots - \mu^p R_p \right) \theta = 0 \tag{11}$$

with

$$\begin{aligned} Q &= T^\top \left(\left(G_q^\top(\theta_k) C_q(\mu_k) G_q(\theta_k) \right)^{-1} \otimes D_q \right) T \\ R_i &= T^\top \left(\left(G_q^\top(\theta_k) C_q(\mu_k) G_q(\theta_k) \right)^{-1} \otimes C_q^{(i)} \right) T \end{aligned}$$

Further reduction in the computational space is possible if a priori knowledge of the equality of certain parameters is available. Introducing the structural constraint matrix S with 0 and 1 entries such that $S\theta = \theta_R$, it is possible to augment the above equations to yield

$$\begin{aligned} Q &= S^\top T^\top \left(\left(G_q^\top(\theta_k) C_q(\mu_k) G_q(\theta_k) \right)^{-1} \otimes D_q \right) TS \\ R_i &= S^\top T^\top \left(\left(G_q^\top(\theta_k) C_q(\mu_k) G_q(\theta_k) \right)^{-1} \otimes C_q^{(i)} \right) TS \end{aligned}$$

and restrict the search for parameter estimates in θ_R with $\dim \theta_R < \dim \theta$.

One way to solve a polynomial eigenvector decomposition problem is by linearization. As a result of linearization, the polynomial eigenvalue problem reduces to a generalized eigenvalue problem. In particular, (11), when subject to symmetry-preserving linearization $\psi(\mu)$ [1] becomes for even p

$$\psi(\mu) = \mathrm{diag}\left(\begin{bmatrix} 0 & I \\ I & R_1 \end{bmatrix}, \begin{bmatrix} 0 & I \\ I & R_3 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & I \\ I & R_{p-1} \end{bmatrix} \right)$$

$$-\mu \mathrm{diag}\left(Q^{-1}, \begin{bmatrix} -R_2 & I \\ I & 0 \end{bmatrix}, \ldots, \begin{bmatrix} -R_{p-2} & I \\ I & 0 \end{bmatrix}, -R_p \right)$$

and for odd p

$$\psi(\mu) = \mathrm{diag}\left(Q, \begin{bmatrix} 0 & I \\ I & R_2 \end{bmatrix}, \begin{bmatrix} 0 & I \\ I & R_4 \end{bmatrix}, \ldots \right)$$

$$-\mu \mathrm{diag}\left(\begin{bmatrix} -R_1 & I \\ I & 0 \end{bmatrix}, \begin{bmatrix} -R_3 & I \\ I & 0 \end{bmatrix}, \ldots, -R_p \right).$$

As the linearized problem has eigenvectors of dimension mp rather than m, the "best" polynomial eigenvector that belongs to the eigenvalue μ becomes the portion v_k of the linearized eigenvector $\psi(\mu)x = 0$ that gives the smallest normalized residual, i.e.

$$v_k = \arg\min_{v_k} \frac{\sum_k |(\Psi(\mu)v)_k|}{\sum_k |v_k|}.$$

With the iterative scheme (10) at hand, a few initial iterations can be used to seed the Levenberg-Marquardt search with appropriate initial values reducing the likelihood of (9) getting stuck in a local minimum.

IV. Model and noise estimation

Contrary to the hidden assumption in the previous section, in a real-world scenario, the true noise structure C_ρ (or equivalently, a noise direction ρ that determines the ratio of input and output noise variances for a unit magnitude noise) is seldom at our disposal. As the final step of the parameter estimation method, we propose means to estimate C_ρ for white noise.

One way to parametrize noise variances, as in (4), is by writing $\sigma_u^2 = \mu \cos^2 \rho$ and $\sigma_y^2 = \mu \sin^2 \rho$ such that $C_q = C_q(\mu, \rho)$. Let $\hat{\theta}$ denote (unit-normalized) estimates obtained with a particular assumption of ρ using (10). Introduce the notations

$$\begin{aligned} \hat{\delta} &= G_q^\top(\hat{\theta}) D_q G_q(\hat{\theta}) \\ \hat{\gamma} &= G_q^\top(\hat{\theta}) C_q(\hat{\mu}, \rho) G_q(\hat{\theta}) \end{aligned}$$

Varying ρ in the range from 0 to $\frac{\pi}{2}$, one can discover the "true" value by minimizing the loss function

$$J(\rho) = \left\| \hat{\delta} - \hat{\gamma} \right\|_F \tag{12}$$

where $\|\cdot\|_F$ denotes the Frobenius norm (a technique called *covariance matching* in [13]) or the "inverted" loss function

$$J(\rho) = \mathrm{trace}\left(\hat{\delta}^{-1} \hat{\gamma} \right) \tag{13}$$

or the so-called Itakura–Saito matrix divergence

$$J(\rho) = \mathrm{trace}(\gamma\delta^{-1}) - \log(\det(\gamma\delta^{-1})) - n \tag{14}$$

where n is the dimension of the square matrices involved. The minimum value for J in the above equations yields the estimated value for ρ.

As an alternative to the two-stage estimation procedure outlined above, a single-stage strategy might theoretically also be employed. Let

$$\begin{aligned}\delta &= G_q^\top(\theta) D_q G_q(\theta) \\ \gamma &= G_q^\top(\theta) C_q(\mu,\, \rho) G_q(\theta)\end{aligned}$$

and introduce the objective functions

$$\begin{aligned}J_1 &= \left(\mathrm{trace}\left(\gamma^{-1}\delta\right) - (q-m)\right)^2 \\ J_2 &= \left(\mathrm{trace}\left(\delta^{-1}\gamma\right) - (q-m)\right)^2 \\ J_3 &= \|\delta-\gamma\|_F^2 = \mathrm{trace}\left\{(\delta-\gamma)^\top(\delta-\gamma)\right\}\end{aligned}$$

Notice that all are functions of θ, μ as well as ρ simultaneously. Due to their nonlinearity, they are likely to exhibit convergence to local minima if started with the wrong initial values.

V. Simulation results

As an introductory example, consider the nonlinear Mackey–Glass chaotic process that models white blood cell production. The process is described by the following state equation

$$x_{0,i+1} = -a x_{0,i} + \frac{b x_{0,i-\tau}}{1 + x_{0,i-\tau}^p} \tag{15}$$

where there exists no external excitation,

$$x_{0,i} = a \quad \text{if} \quad 0 \leqq i \leqq \tau$$

$$a = 0.9 \quad b = 0.2 \quad \tau = 17 \quad p = 10$$

The relationship (15) given in explicit form can be reformulated in implicit form

$$\begin{aligned}x_{0,i+1} x_{0,i-\tau}^p + a x_{0,i} x_{0,i-\tau}^p &+ \\ x_{0,i+1} + a x_{0,i} - b x_{0,i-\tau} &= 0\end{aligned}$$

A sequence of $N = 1000$ data are observed with a noise of $\sigma^2 = 0.05$. Notice the equality constraint on certain parameter coefficients expressible with the structural constraint matrix

$$S = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

with the original model parameter vector θ (without repeated coefficients) relating to the extended vector $\bar{\theta}$ (with repeated coefficients) via $S\bar{\theta} = \theta$. As a result of a simulation with $M = 100$ runs, $\hat{a} = 0.9147 \pm 0.0124$ and $\hat{b} = -0.1653 \pm 0.0334$, that reproduce the original process fairly accurately.

Let us now draw our attention to an artificial yet relatively complex process described by the nonlinear relationship

$$\begin{aligned}y_{0,i} &= p_1 y_{0,i-1} + p_2 y_{0,i-2} + p_3 u_{0,i-1} + \\ &+ p_4 u_{0,i-1}^2 + p_5 y_{0,i-1} y_{0,i-2} \\ &+ p_6 u_{0,i-1} y_{0,i-1}\end{aligned}$$

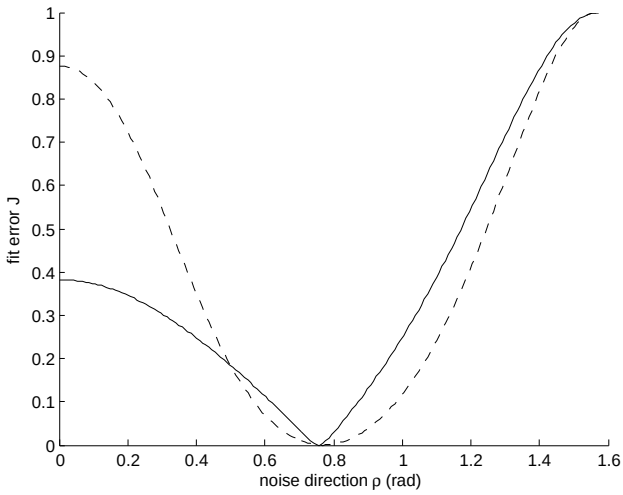

Fig. 3. Discovering noise direction by successive estimation over an angle range (continuous: Frobenius (12); dashed: Itakura–Saito (14) distance measure).

comprising of both linear and polynomial terms as well as cross-correlating terms. The true parameter values are set to

$$\begin{aligned}p_1 &= 1.5 & p_2 &= -0.7 & p_3 &= 1 \\ p_4 &= -0.3 & p_5 &= -0.05 & p_6 &= 0.1\end{aligned}$$

and a configuration of $N = 500$ samples, $q = 6$, $\sigma_u = 0.01$ and $\sigma_y = 0.01$ is set up.[1] Next, a Monte-Carlo simulation of $M = 100$ runs of the outlined nonlinear extension to the generalized Koopmans–Levin method has been carried out with a known noise structure. Table I analyzes the consistency of the estimation scheme.

Figure 3 shows how to discover noise covariance structure (i.e. the noise "direction" ρ) by minimizing the distance or divergence measures (12) and (13) over an interval to arrive at estimates for all parameters. In accordance, a different series of Monte-Carlo experiments have been conducted, in which for each independent simulation run a noise structure discovery step shown in Figure 3 has been carried out, with model parameters and noise magnitude estimated for each potential value of ρ, and the distance measure evaluated. The comprehensive results, in which all model and noise parameters are estimated, are shown in Table II.

For the sake of comparison, an instrumental variable scheme, based on the bias-compensating least-squares technique for nonlinear polynomial systems (NBCLS) has been included in the table. The scheme minimizes the objective function [5]

$$J = \left\|d_{IV} - c_{IV} - (D_{IV} - C_{IV})\bar{\theta}\right\| \tag{16}$$

where

$$\bar{\theta} = (D_{IV} - C_{IV})^\dagger (d_{IV} - c_{IV})$$

with D_{IV} and d_{IV} being the (rectangular) covariance matrices of the regressor vector and the output vector, respectively, w.r.t. so-called instruments and $C_{IV}(\mu,\, \rho)$ and $c_{IV}(\mu,\, \rho)$

[1] Simulation examples are available as part of [4].

nonlinearity	true value	$N = 500$			$N = 1000$			$N = 2000$		
$\hat{y}_{0,i-1}$	-1.5	-1.5001	±	0.0039	-1.4998	±	0.0030	-1.5000	±	0.0020
$\hat{y}_{0,i-2}$	0.7	0.7001	±	0.0034	0.6998	±	0.0024	0.7001	±	0.0014
$\hat{u}_{0,i-1}$	-1.0	-1.0009	±	0.0197	-1.0008	±	0.0135	-1.0001	±	0.0093
$\hat{u}^2_{0,i-1}$	0.3	0.3256	±	0.1066	0.2919	±	0.0649	0.3004	±	0.0537
$\hat{y}_{0,i-1}\hat{y}_{0,i-2}$	0.05	0.0502	±	0.0137	0.0500	±	0.0077	0.0497	±	0.0051
$\hat{u}_{0,i-1}\hat{y}_{0,i-1}$	-0.1	-0.0986	±	0.0413	-0.0988	±	0.0259	-0.0992	±	0.0172
$\mu \cdot 10^{-3}$	0.2	0.1958	±	0.0118	0.1978	±	0.0086	0.1982	±	0.0067

TABLE I

CONSISTENCY ANALYSIS OF PARAMETER ESTIMATES WITH $q = 6$ AND KNOWN NOISE STRUCTURE.

nonlinearity	NBCLS (16)			Frobenius norm (12)			"inverted" (13)			Itakura–Saito (14)		
$\hat{y}_{0,i-1}$	-1.4996	±	0.0045	-1.5002	±	0.0039	-1.4997	±	0.0044	-1.4998	±	0.0036
$\hat{y}_{0,i-2}$	0.6998	±	0.0038	0.7002	±	0.0032	0.6997	±	0.0035	0.7000	±	0.0028
$\hat{u}_{0,i-1}$	-1.0015	±	0.0096	-0.9998	±	0.0182	-1.0021	±	0.0221	-0.9980	±	0.0198
$\hat{u}^2_{0,i-1}$	0.3258	±	0.3173	0.3101	±	0.0953	0.2928	±	0.1053	0.3004	±	0.0898
$\hat{y}_{0,i-1}\hat{y}_{0,i-2}$	0.0500	±	0.0040	0.0492	±	0.0114	0.0496	±	0.0104	0.0505	±	0.0102
$\hat{u}_{0,i-1}\hat{y}_{0,i-1}$	-0.1058	±	0.0477	-0.1048	±	0.0391	-0.1017	±	0.0309	-0.1010	±	0.0338
$\sigma^2_y \left[10^{-3}\right]$	0.0984	±	0.0153	0.1017	±	0.0131	0.0999	±	0.0099	0.1011	±	0.0102
$\sigma^2_u \left[10^{-3}\right]$	0.1021	±	0.0406	0.0915	±	0.0138	0.0944	±	0.0126	0.0911	±	0.0102

TABLE II

SIMULATION RESULTS WITH UNKNOWN NOISE STRUCTURE AND VARIOUS METHODS TO MATCH COVARIANCE MATRICES.

being the corresponding computed (rectangular) noise covariance matrices. Instruments include the regressor vector as well as past observations not in the system model. The notation $M^\dagger$ stands for the Moore-Penrose pseudoinverse $\left(M^\top M\right)^{-1} M^\top$.

Even though the proposed method exhibits somewhat but not substantially larger variance for some parameters, NBCLS produces strikingly high variance for parameters $\hat{u}^2_{0,i-1}$, $\hat{u}_{0,i-1}\hat{y}_{0,i-1}$ and σ^2_u, making estimates for these parameters of little use.

VI. CONCLUSION

After a brief description of the Koopmans–Levin method and its generalization for linear systems, we introduced a nonlinear extension by modifying the objective function and the iteration scheme of the linear generalization. As a result, the original dependency on the model parameters θ has been extended with dependency on the noise magnitude μ. Consequently, the iterative problem is tackled efficiently by solving a polynomial rather than a generalized eigenvalue problem. Next, an optimization scheme of minimizing an error term over an angle range has been shown to give a noise structure estimate thus yielding estimates for all model and noise parameters. Finally, objective functions that encompass all model and noise parameters have been proposed, which can be fed to direct search methods to produce parameter estimates.

ACKNOWLEDGMENTS

This work has been supported by the fund of the Hungarian Academy of Sciences for control research and the Hungarian National Research Fund (grant number T68370).

REFERENCES

[1] Efstathios N. Antoniou and Stavros Vologiannidis. Linearizations of polynomial matrices with symmetries and their applications. *Electronic Journal of Linear Algebra*, 15:107–114, February 2006.

[2] Mei Hong, Torsten Söderström, Umberto Soverini, and Roberto Diversi. Comparison of three Frisch methods for errors-in-variables identification. In *Proc. of 17th IFAC World Congress*, pages 420–425, Seoul, Korea, July 2008.

[3] Mei Hong, Torsten Soderstrom, and Wei Xing Zheng. A simplified form of the bias-eliminating least squares method for errors-in-variables identification. *Automatic Control, IEEE Transactions on*, 52(9):1754–1756, September 2007.

[4] Levente Hunyadi. Fræser: a MatLab framework for estimating errors-in-variables systems. Available online: `http://fraeser.sourceforge.net`, June 2009.

[5] Tomasz Larkowski, Jens G. Linden, Benoit Vinsonneau, and Keith J. Burnham. Novel algorithms based on conjunction of the Frisch scheme and extended compensated least squares. In *International Conference on Control*, Manchester, UK, 2008.

[6] Morris J. Levin. Estimation of a system pulse transfer function in the presence of noise. In *Proc. of Joint Automatic Control Conference*, pages 452–458, 1963.

[7] Ivan Markovsky, Alexander Kukush, and Sabine Van Huffel. On errors-in-variables with unknown noise variance ratio. In *Proc. of 14th IFAC Symposium on System Identification*, pages 172–177, March 2006.

[8] Torsten Söderström. Errors-in-variables methods in system identification. *Automatica*, 43:939–958, 2007.

[9] Stéphane Thil, Marion Gilson, and Hugues Garnier. On instrumental variable-based methods for errors-in-variables model identification. In *Proc. of 17th IFAC World Congress*, pages 420–425, Seoul, Korea, July 2008.

[10] Stéphane Thil, Mei Hong, Torsten Söderström, Marion Gilson, and Hugues Garnier. Statistical analysis of a third-order cumulants based algorithm for discrete-time errors-in-variables identification. In *Proc. of 17th IFAC World Congress*, pages 420–425, Seoul, Korea, July 2008.

[11] István Vajk. Identification methods in a unified framework. *Automatica*, 41(8):1385–1393, 2005.

[12] István Vajk and Jenő Hetthéssy. Identification of nonlinear errors-in-variables models. *Automatica*, 39:2099–2107, 2003.

[13] István Vajk and Jenő Hetthéssy. Efficient estimation of errors-in-variables models. In *17th IFAC World Congress*, pages 1384–1389, Seoul, Korea, July 2008.